A GENERALIZED VERSION OF THE
PERTURBATION-BASED STOCHASTIC FINITE DIFFERENCE METHOD
FOR ELASTIC BEAMS

Abstract – The main idea here is to demonstrate a stochastic computational technique consisting of the generalized stochastic perturbation technique using the Taylor expansions of the random variables and the classical Finite Difference Method based on the regular grids. As it is documented by the computational illustrations, it is possible to determine using this approach also higher probabilistic moments for any random dispersion of input variables unlike in the second order second moment technique worked out before. A numerical algorithm is implemented here using the straightforward partial differentiation of the hierarchical equations with respect to the random input quantity and further symbolic computations of probabilistic moments and characteristics by the system MAPLE.

Keywords: stochastic perturbation technique, finite difference method, response function method, elastic Euler-Bernoulli beam, elastic beam on Winkler foundation

1. INTRODUCTION

The Finite Difference Method (FDM) implementation (Liszka and Orkisz, 1980, Wasow and Forsythe, 1959) plays an important role in computational engineering in all those cases, where the additional differential equations (Collatz, 1960) (ordinary or partial) may not be solved straightforwardly – i.e. in the heat transfer (Minkowycz et al., 1988), electro-magnetics (Taflove, 1998) and geodynamics; even in elasto-statics, as it is demonstrated here. So that, it seems to be natural that this method is extended towards its new stochastic versions for some real systems with random parameters solved before using the traditional FDM in deterministic cases. One of such an extension method is the generalized perturbation-based stochastic technique, where the Taylor series expansions of all random quantities lead to the system of equilibrium equations of the ascending order. This method was employed before for different stochastic extensions for the Finite Element Method (Kamiński, 2007, Kleiber and Hien, 1992), Boundary Element Method as well as even Finite Difference Method (Kamiński, 2001) (according to the second order second moment approach) but this implementation for the first time enables for (1) any order stochastic expansion, (2) any probability density function of the random input variable, (3) parametric study with respect to the perturbation parameter and coefficient of variation for the random input as well as (4) analytical derivations of most of discrete hierarchical equations implemented in the symbolic package MAPLE (the other systems like freeware Scilab are also available of course).
The major difference in the comparison to the stochastic versions of the FEM and BEM are the necessity of double differentiation – with respect to the space variable discretized by Δx (4th order derivatives transformed to the finite differences) as well as with respect to the input random variables. Fortunately, since an application of the symbolic calculus, this second differentiation is performed analytically for any available derivatives orders but in the case of a general computer program those derivatives must be implemented into it in a form of the ready-to-use-formulas (or we need to assure the interoperability with the MAPLE environment). The remaining implementation issues are almost the same like in the case of the SFEM and the SBEM, but in a further perspective a comparison with the other stochastic methods like polynomial chaos expansions (Ghanem and Spanos, 1991) or Monte-Carlo simulations would be interesting (Hurtado and Barbat, 1998).

The stochastic implementation of the Finite Difference Method is displayed and discussed here on the example of the well-known fourth order ordinary differential equations relevant to the elastic homogeneous and isotropic beams with and without the elastic single-parameter foundation. Although the general computer program is written in the internal language of the symbolic computing environment MAPLE, the algorithm has a general character and the grid applied to any beam may be essentially densified without any larger programming, where a formation and the solution of the ascending order hierarchical equations typical for the perturbation-based methodology will remain the same. Contrary to the previous second order second moment (SOSM) technique, now it is possible to compute any order moments (up to the fourth order here) for practically any value of the standard deviation for the random input. This methodology will be further extended towards 2 and 3-dimensional applications, also for transient problems with the random coefficients.

2. CLASSICAL VERSION OF THE FINITE DIFFERENCE METHOD FOR THE ELASTIC BEAMS

Let us consider the following ordinary fourth order differential equation for the linear elastic isotropic and statistically homogeneous beam exposed to the transversally distributed load \( q(x) \)

\[
\frac{d^2}{dx^2} \left( E(x) J(x) \frac{d^2 w(x)}{dx^2} \right) = q(x)
\]  

(2.1)

fulfilling the typical boundary conditions applicable in engineering theories of the elastic beams.

\[ \text{Figure 1. Elastic beam subjected to the transversal load } q(x) \]

Let us also note that we neglect further the influence of the longitudinal and transversal forces as well as we reduce the analysis to the small deflections, which significantly simplifies the final form of this equation. Let us divide the entire domain of the length \( l \) into \( n \) equidistant
sub-domains with the length $\Delta x$. So that, for the $i$th point of this discretization we adopt the following notation:

$$E(i\Delta x) = E_i, \ J(i\Delta x) = J_i, \ q(i\Delta x) = q_i.$$  \hspace{1cm} (2.2)

On the other hand, it follows the series of the approximations for the derivatives of the ascending order with some finite differences like that:

- **first derivative**

$$\left( \frac{\Delta w}{\Delta x} \right)_i = \frac{w_{i+1} - w_{i-1}}{2\Delta x}, \hspace{1cm} (2.3)$$

- **second derivative**

$$\left( \frac{\Delta^2 w}{\Delta x^2} \right)_i = \frac{w_{i+1} - 2w_i + w_{i-1}}{\Delta x^2}, \hspace{1cm} (2.4)$$

- **third derivative**

$$\left( \frac{\Delta^3 w}{\Delta x^3} \right)_i = -\frac{w_{i+2} - 2w_{i+1} - 2w_{i-1} + w_{i-2}}{2\Delta x^3}, \hspace{1cm} (2.5)$$

- **fourth derivative**

$$\left( \frac{\Delta^4 w}{\Delta x^4} \right)_i = \frac{w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}}{\Delta x^4}.$$ \hspace{1cm} (2.6)

Introducing for the fourth order derivatives in eqn (2.1) the above relation one can obtain for the $i$th point of this grid that

$$E_{i-1}J_{i-1}w_{i-2} - 2(E_{i-1}J_{i-1} + E_iJ_i)w_{i-1} + (E_{i-1}J_{i-1} + 4E_iJ_i + E_{i+1}J_{i+1})w_i +$$

$$- 2(E_iJ_i + E_{i+1}J_{i+1})w_{i+1} + E_{i+1}J_{i+1}w_{i+2} = q_i\Delta x^4 \hspace{1cm} (2.7)$$

The particular case of $E(x)J(x) =$ const. $= EJ$ enables to transform eqn (2.7) into the formula:

$$w_{i-2} - 4w_{i-1} + 6w_i - 4w_{i+1} + w_{i+2} = \frac{q_i\Delta x^4}{EJ}. \hspace{1cm} (2.8)$$

The situation changes only slightly when the elastic beam rests on the single parameter elastic foundation (known as the Winkler foundation). Let us assume that this foundation has a homogeneous character and is completely characterized with the compliance coefficient $k$. Therefore, the equilibrium equation including the deflection $w(x)$ possesses a single extra component at the right hand side. There holds

$$\frac{d^2}{dx^2} \left( E(x)J(x) \frac{d^2 w(x)}{dx^2} \right) = -kw(x) + q(x). \hspace{1cm} (2.9)$$

Inserting, as previously, the additional formulas for the derivatives, which include finite differences expressed by the discrete values of the function $w(x)$ in the neighborhood of the given $i$th point of the grid, it is obtained that

$$E_{i-1}J_{i-1}w_{i-2} - 2(E_{i-1}J_{i-1} + E_iJ_i)w_{i-1} + (E_{i-1}J_{i-1} + 4E_iJ_i + E_{i+1}J_{i+1} + k\Delta x^4)w_i +$$

$$- 2(E_iJ_i + E_{i+1}J_{i+1})w_{i+1} + E_{i+1}J_{i+1}w_{i+2} = q_i\Delta x^4 \hspace{1cm} (2.10)$$
The entire situation and the FDM discretization is schematically presented in Fig. 2 below.

Figure 2. Finite Difference Method discretization of the elastic beam

A reduction for the case $E(x)J(x)=$const.$=EJ$ leads to the following equation:

$$w_{i-2} - 4w_{i-1} + \left( 6 + \frac{k\Delta x^4}{EJ} \right) w_i - 4w_{i+1} + w_{i+2} = \frac{q_i \Delta x^4}{EJ}.$$  \hspace{1cm} (2.11)

3. PERTURBATION-BASED NTH ORDER APPROACH TO THE FINITE DIFFERENCE ANALYSIS

Let us denote the corresponding random vector of the problem by $b(x)$, with probability density function $p(b)$. Therefore, the $m$th order central probabilistic moment of any random function of this parameter, namely $F(b)$, is defined as

$$\mu_m(F(b)) = \int_{-\infty}^{+\infty} (F(b) - E[F(b)])^m p(b) db.$$ \hspace{1cm} (3.1)

The basic idea of the stochastic perturbation approach follows the classical expansion idea and is based on an approximation of all input variables and the state functions of the problem via truncated Taylor series about their spatial expectations in terms of a parameter $\varepsilon>0$. For example, in the case of a random deflection, the $n$th order truncated expansion may be written as

$$w(b) = w^0(b^0) + \sum_{k=1}^{n} \frac{\varepsilon^k}{k!} (\Delta b)^k \frac{\partial^k w(b)}{\partial b^k},$$ \hspace{1cm} (3.2)

where

$$\varepsilon \Delta b = \varepsilon (b - b^0)$$ \hspace{1cm} (3.3)

is the first variation of $b$ about its expected value and, similarly,

$$\varepsilon^2 (\Delta b)^2 = \varepsilon^2 (b - b^0)^2$$ \hspace{1cm} (3.4)

is the second variation of $b$ around its expected value, where $n$th order variation can be expressed accordingly. Traditionally, the stochastic perturbation approach to all the physical problems is entered by the respective perturbed equations of the zeroth, first and successively higher orders being a modification of the variational integral formulation. Hence, there holds
one zeroth order partial differential equation
\[
E^0 J^0 \frac{d^4}{dx^4}(w^0(x)) = q^0, \tag{3.5}
\]

first order partial differential equation
\[
\frac{\partial E}{\partial b} J^0 \frac{d^4}{dx^4}(w^0(x)) + E^0 \frac{\partial J}{\partial b} \frac{d^4}{dx^4}(w^0(x)) + E^0 J^0 \frac{d^4}{dx^4}(\partial w(x)) = \frac{\partial q}{\partial b}, \tag{3.6}
\]

and

second order partial differential equation
\[
\frac{\partial^2 E}{\partial b^2} J^0 \frac{d^4}{dx^4}(w^0(x)) + E^0 \frac{\partial^2 J}{\partial b^2} \frac{d^4}{dx^4}(w^0(x)) + E^0 J^0 \frac{d^4}{dx^4}(\partial^2 w(x)) +
\]
\[
2 \left( \frac{\partial E}{\partial b} \frac{\partial J}{\partial b} \frac{d^4}{dx^4}(w^0(x)) + E^0 \frac{\partial J}{\partial b} \frac{d^4}{dx^4}(\partial w(x)) + \frac{\partial E}{\partial b} J^0 \frac{d^4}{dx^4}(\partial w(x)) \right) = \frac{\partial^2 q}{\partial b^2}. \tag{3.7}
\]

Quite similarly one can derive the ascending order partial differential equations for the elastic beam on the elastic foundation starting from a relation (2.9)

one zeroth order partial differential equation
\[
E^0 J^0 \frac{d^4}{dx^4}(w^0(x)) = -k^0 w^0(x) + q^0, \tag{3.8}
\]

first order partial differential equation
\[
\frac{\partial E}{\partial b} J^0 \frac{d^4}{dx^4}(w^0(x)) + E^0 \frac{\partial J}{\partial b} \frac{d^4}{dx^4}(w^0(x)) + E^0 J^0 \frac{d^4}{dx^4}(\partial w(x)) = \frac{\partial q}{\partial b} - \left( \frac{\partial k}{\partial b} w^0(x) + k^0 \frac{\partial w(x)}{\partial b} \right) \tag{3.9}
\]

as well as

second order partial differential equation
\[
\frac{\partial^2 E}{\partial b^2} J^0 \frac{d^4}{dx^4}(w^0(x)) + E^0 \frac{\partial^2 J}{\partial b^2} \frac{d^4}{dx^4}(w^0(x)) + E^0 J^0 \frac{d^4}{dx^4}(\partial^2 w(x)) +
\]
\[
2 \left( \frac{\partial E}{\partial b} \frac{\partial J}{\partial b} \frac{d^4}{dx^4}(w^0(x)) + E^0 \frac{\partial J}{\partial b} \frac{d^4}{dx^4}(\partial w(x)) + \frac{\partial E}{\partial b} J^0 \frac{d^4}{dx^4}(\partial w(x)) \right) =
\]
\[
\frac{\partial^2 q}{\partial b^2} - \left( \frac{\partial^2 k}{\partial b^2} w^0(x) + 2 \frac{\partial k}{\partial b} \frac{\partial w(x)}{\partial b} + k^0 \frac{\partial^2 w(x)}{\partial b^2} \right) \tag{3.10}
\]

Higher order equations derivation proceeds quite similarly – by systematic differentiation until the nth order equation is recovered. Having solved those equations for \(w^0(x)\) and their higher orders, respectively, (specifically its partial derivatives \(w.r.t.\) random input within all discrete points of the grid), we derive the expressions for the expected values and the other moments of the elastic beam deflections. In order to calculate the expected values and higher order probabilistic moments of \(w(x;b)\), the same Taylor expansion is employed for the definitions of its probabilistic moments; there holds
\[ E[w(b)] = \int_{\infty}^{\infty} w(b) p(b) db = \int_{\infty}^{\infty} w^0(b^0) + \sum_{k=1}^{n} \frac{\varepsilon^k}{k!} (\Delta b)^k \frac{\partial^k w(b)}{\partial b^k} \] \end{equation}

If there is a high random dispersion in the input random variable and the symmetric probability density function is chosen, then the generalized expansion simplifies to

\begin{equation}
E[w(b)] = w^0(b^0) + \sum_{k=1}^{n} \frac{\varepsilon^{2k}}{(2k)!} \frac{\partial^{2k} w(b)}{\partial b^{2k}} \mu_{2k}(b),
\end{equation}

where \( \mu_{2k}(b) \) denote 2\( k \)th order probabilistic moment of the variable \( b \). When the probability density function is defined as the Gaussian one with the standard deviation \( \sigma \), we obtain additionally

\begin{equation}
\mu_{2k+1}(b) = 0, \quad \mu_{2k}(b) = (2k-1)! \sigma^{2k}(b).
\end{equation}

Using such an extension of the random input, a desired efficiency of the expected values can be achieved by the appropriate choice of the perturbation parameter and maximum order corresponding to the particular input probability density function type, probabilistic moments interrelations, acceptable error of the computations, etc. This choice can be made reasonably by the comparative studies with the Monte-Carlo simulations or theoretical results obtained by the direct (i.e. symbolic) integration. Consequently, \( m \)th order probabilistic moment for the structural response function in the \( n \)th order stochastic Taylor expansion is introduced as

\begin{equation}
\mu_m(w(b)) = \int_{\infty}^{\infty} \left( w^0(b^0) + \sum_{k=1}^{n} \frac{\varepsilon^k}{k!} (\Delta b)^k \frac{\partial^k w(b)}{\partial b^k} - E[w(b)] \right)^m p(b) db = \int_{\infty}^{\infty} \left( \sum_{k=1}^{n} \frac{\varepsilon^k}{k!} (\Delta b)^k \frac{\partial^k w(b)}{\partial b^k} \right)^m p(b) db.
\end{equation}

Taking the first few components only, one can demonstrate that the relevant expansions for the 3\( r d \) and 4\( r d \) order moments (Kamiński, 2007) equal to

\begin{equation}
\mu_3(w(b)) = \frac{3}{2} \varepsilon^4 \mu_4(b) \left( \frac{\partial w}{\partial b} \right)^2 \frac{\partial^3 w}{\partial b^2} + \frac{1}{8} \varepsilon^6 \mu_6(b) \left( \frac{\partial^2 w}{\partial b^2} \right)^3
\end{equation}

and

\begin{equation}
\mu_4(w(b)) = \frac{3}{2} \varepsilon^4 \mu_4(b) \left( \frac{\partial w}{\partial b} \right)^4 + \frac{3}{2} \varepsilon^6 \mu_6(b) \left( \frac{\partial w}{\partial b} \frac{\partial^2 w}{\partial b^2} \right)^2 + \frac{1}{16} \varepsilon^8 \mu_8(b) \left( \frac{\partial w}{\partial b} \right)^3 \left( \frac{\partial^2 w}{\partial b^2} \right)^4.
\end{equation}

The discrete equations for the Stochastic Finite Difference Method build upon the above equations for the perturbation-based analysis are essentially different in the case of an elastic beam with and without foundation beneath it. Putting here \( k=0 \) returns various order relations as for example

- zeroth order relation

\[ w_{i-2}^0 - 4w_{i-1}^0 + 6w_i^0 - 4w_{i+1}^0 + w_{i+2}^0 = \frac{q_i \Delta x^4}{E^0 J^0}, \]

- \( n \)th order relation

\[ \frac{\partial^n w_{i-2}}{\partial b^n} - 4 \frac{\partial^n w_{i-1}}{\partial b^n} + 6 \frac{\partial^n w_i}{\partial b^n} - 4 \frac{\partial^n w_{i+1}}{\partial b^n} + \frac{\partial^n w_{i+2}}{\partial b^n} = \frac{\partial^n}{\partial b^n} \left( \frac{q_i \Delta x^4}{E J} \right), \]
so that the uniformly distributed load gives here nonzero equations of up to the first order only, random length results in up to the fourth order equations, whereas the cross-sectional and/or material randomness brings here an infinite number of equations. Looking for the perturbation-based SFDM equations for the beams on the elastic foundation one can receive

- zeroth order equations

\[ w_{i-2}^0 - 4w_{i-1}^0 + \left( 6 + \frac{k^0 (\Delta x^0)^2}{E^0 J^0} \right) w_i^0 - 4w_{i+1}^0 + w_{i+2}^0 = \frac{q_i^0 (\Delta x^0)^4}{E^0 J^0}, \]  

(3.19)

- the \( n \)th order equations

\[ \frac{\partial^n w_{i-2}}{\partial b^n} - 4 \frac{\partial^n w_{i-1}}{\partial b^n} + 6 \frac{\partial^n w_i}{\partial b^n} + \sum_{p=1}^{n} \binom{n}{p} \frac{\partial^p}{\partial b^p} \left( \frac{k\Delta x^4}{EJ} \right) \frac{\partial^{n-p} w_i}{\partial b^{n-p}} - 4 \frac{\partial^n w_{i+1}}{\partial b^n} + \frac{\partial^n w_{i+2}}{\partial b^n} = \frac{\partial^n}{\partial b^n} \left( \frac{q_i \Delta x^4}{EJ} \right) \]  

(3.20)

Let us also note that \( n-1 \) of those equations may be generated automatically from the zeroth order formula using the symbolic computations systems like MAPLE in the computational illustrations below.

4. Computational illustrations

4.1. Deflection of the linear elastic beam with linearly varying cross-sectional area

Let us determine the first four probabilistic moments for the cantilever beam with linearly varying cross-sectional area under the constant distributed load \( q=1.0 \) kN/m. Young modulus of this beam is introduced as the input Gaussian random variable, where the expected value is given as \( E[E]=206,01 \) GPa, a grid for this structures consisting of 6 elements with the constant length equal to \( \Delta x=0.1 \) m is proposed below (see Pietrzak et al., 1986 for deterministic test).
The following central finite difference beam equations hold true in this particular case:

\[ J_0 w_{-1} - 2(J_0 + J_1)w_0 + (J_0 + 4J_1 + J_2)w_1 - 2(J_1 + J_2)w_2 + J_2 w_3 = \frac{q\Delta x^4}{E}, \]

\[ J_1 w_0 - 2(J_1 + J_2)w_1 + (J_1 + 4J_2 + J_3)w_2 - 2(J_2 + J_3)w_3 + J_3 w_4 = \frac{q\Delta x^4}{E}, \]

\[ J_2 w_1 - 2(J_2 + J_3)w_2 + (J_2 + 4J_3 + J_4)w_3 - 2(J_3 + J_4)w_4 + J_4 w_5 = \frac{q\Delta x^4}{E}, \]

\[ J_3 w_2 - 2(J_3 + J_4)w_3 + (J_3 + 4J_4 + J_5)w_4 - 2(J_4 + J_5)w_5 + J_5 w_6 = \frac{q\Delta x^4}{E}, \]

\[ J_4 w_3 - 2(J_4 + J_5)w_4 + (J_4 + 4J_5 + J_6)w_5 - 2(J_5 + J_6)w_6 + J_6 w_7 = \frac{q\Delta x^4}{E}, \]

\[ J_5 w_4 - 2(J_5 + J_6)w_5 + (J_5 + 4J_6 + J_7)w_6 - 2(J_6 + J_7)w_7 + J_7 w_8 = \frac{q\Delta x^4}{E}. \]

Introducing the fictitious nodes as well as using the kinematic boundary conditions we obtain

\[ (2J_0 + 4J_1 + J_2)w_1 - 2(J_1 + J_2)w_2 + J_2 w_3 = \frac{q\Delta x^4}{E}, \]

\[ -2(J_1 + J_2)w_1 + (J_1 + 4J_2 + J_3)w_2 - 2(J_2 + J_3)w_3 + J_3 w_4 = \frac{q\Delta x^4}{E}, \]
\[ J_2 w_1 - 2(J_2 + J_3)w_2 + (J_2 + 4J_3 + J_4)w_3 - 2(J_3 + J_4)w_4 + J_4w_5 = \frac{q\Delta x^4}{E}, \]

\[ J_3 w_2 - 2(J_3 + J_4)w_3 + (J_3 + 4J_4 + J_5)w_4 - 2(J_4 + J_5)w_5 + J_5w_6 = \frac{q\Delta x^4}{E}, \]

\[ J_4 w_3 - 2(J_4 + J_5)w_4 + (J_4 + 4J_5)w_5 - 2J_5w_6 = \frac{q\Delta x^4}{E}, \]

\[ 2J_5w_4 - 4J_5w_5 + 2J_5w_6 = \frac{q\Delta x^4}{E}. \]

This system of linear equations may be used for further analytical partial differentiation with respect to the random input variable and a formation of the higher order equations. The symbolic computations package MAPLE, v. 11, is employed for a derivation of up to the tenth order equilibrium perturbation-based equations and determination of the expected values (each time for the maximum deflection at the cantilever end) according to the 2\textsuperscript{nd}, 4\textsuperscript{th}, 6\textsuperscript{th}, 8\textsuperscript{th} and 10\textsuperscript{th} approaches (Fig. 4, where the additional order results are marked with the additional numbers), the computations of the standard deviations and variances in the framework of the 2\textsuperscript{nd}, 4\textsuperscript{th} and 6\textsuperscript{th} order theories (Figs. 5 and 6) as well as finally, a derivation of the third (Fig. 7) and the fourth central probabilistic moments (Fig. 8). All those computations are performed with respect to the perturbation parameter \( \varepsilon \) belonging to the interval \([0.8,1.2]\) and, secondly, the coefficient of variation of the randomized Young modulus \( \alpha(b) \) – standard deviation vs. the expected value - for the beam taken from the interval \([0.0,0.3]\). The problems with this coefficient equal to 0 are adequate to the deterministic tests, of course, so that they can be treated as evaluation tests being quite insensitive to the perturbation parameter at all. Generally, it is clear that the particular values of all probabilistic characteristics increase together with the order of the method, and, furthermore, that the difference between the results of the ascending order methods systematically decrease. It is important to mention that for the clarity of the presentation those differences are visualized using the largest perturbation parameter value – 1.2 remaining not so transparent for its value taken in most of the engineering computations as 1.0. Anyway, even for the most extremum combinations of the problem parameters, the differences between the neighboring orders results are smaller than a single percent of the computed deflection value.

Further, as one may recognize, an increase of the perturbation parameter and, at the same time, the additional increase of the coefficient of variation leads each time to the nonlinear increase of the probabilistic moment being computed. The impact of the perturbation parameter is very small in the second order approach and becomes more and more important together with an increase of the theory having a similar importance as variation coefficient for the tenth order technique. Neglecting the theory order, the coefficient of variation of the input Young modulus being randomized here is more decisive for the parameter variability of the deflections probabilistic characteristics.
Figure 4. Expected values as the function of the perturbation parameter and input coefficient of variation (2\textsuperscript{nd}, 4\textsuperscript{th}, 6\textsuperscript{th}, 8\textsuperscript{th} and 10\textsuperscript{th} orders)

Figure 5. Standard deviations as the function of the perturbation parameter and input coefficient of variation (2\textsuperscript{nd} and 4\textsuperscript{th} orders)
Figure 6. Variances as the function of the perturbation parameter and input coefficient of variation (2\textsuperscript{nd} and 4\textsuperscript{th} orders)

Figure 7. 3\textsuperscript{rd} central probabilistic moments as the function of the perturbation parameter and input coefficient of variation
A comparison of the deflections in a deterministic case together with the expected values computed for the standard deviation of it equal to 10% of the Young modulus expected value is given below. As it is clear from this table, expected values are larger than the corresponding deterministic values, which of course reflects the perturbation-based formula for expectations, see eqn (19), where deterministic value is the first component, while the second one remains always positive.

**Tab. No. 1. Deflections and their expected values for the grid points in test no 1**

<table>
<thead>
<tr>
<th>Node number</th>
<th>( w ) [m]</th>
<th>( E[w] ) [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.828443 ( \times ) 10^-3</td>
<td>0.816330 ( \times ) 10^-5</td>
</tr>
<tr>
<td>2</td>
<td>0.311575 ( \times ) 10^-4</td>
<td>0.310456 ( \times ) 10^-4</td>
</tr>
<tr>
<td>3</td>
<td>0.664578 ( \times ) 10^-4</td>
<td>0.664655 ( \times ) 10^-4</td>
</tr>
<tr>
<td>4</td>
<td>0.111347 ( \times ) 10^-3</td>
<td>0.111559 ( \times ) 10^-3</td>
</tr>
<tr>
<td>5</td>
<td>0.162304 ( \times ) 10^-3</td>
<td>0.162775 ( \times ) 10^-3</td>
</tr>
<tr>
<td>6</td>
<td>0.215525 ( \times ) 10^-3</td>
<td>0.216275 ( \times ) 10^-3</td>
</tr>
</tbody>
</table>

4.2. Deflection of the linear elastic beam resting on the elastic Winkler foundation

The second case study deals with quite a similar cantilever beam with the constant cross-sectional area however, where \( J = 1.71 \times 10^{-6} \text{m}^4 \), resting now on the elastic foundation characterized by the coefficient \( k = 5 \times 10^{7} \frac{N}{\text{m}^3} \). The external distributed load is constant along this beam \( q = 10000 \frac{N}{m} \) for \( l = 3.0 \text{ m} \); the structure is discretized with \( \Delta x = 0.5 \text{ m} \), and, analogously to the previous case, the Young modulus is the input Gaussian random variable.
with the expectation equal to $E = 206.01 \text{ GPa}$ (see the deterministic test in Pietrzak et al., 1986). The fundamental zeroth order SFDM equation has the following form:

$$w_{i-2} - 4w_{i-1} + 14.8709w_i - 4w_{i+1} + w_{i+2} = 0.0177416$$  \hspace{1cm} (4.3)

with the following boundary conditions:

$$
\begin{align*}
  w_0 &= 0 \\
  w_{i-1} &= w_i \\
  w_7 &= 2w_6 - w_5 \\
  w_8 &= 4w_6 - 4w_5 + w_4
\end{align*} \hspace{1cm} (4.4)
$$

where the nodes 7. and 8. are fictitious, as before. A comparison of the deflections in a deterministic case together with the expected values computed for the standard deviation of it equal to 10% of the Young modulus expected value is given below. As it is clear from this table, expected values are larger than the corresponding deterministic values, which of course once more reflect the perturbation-based formula for the expectations.

<p>| Tab. No. 2. Deflections and their expected values for the grid points in test no 2 |
|---------------------------------|----------------|----------------|</p>
<table>
<thead>
<tr>
<th>Node number</th>
<th>$w$ [m]</th>
<th>$E[w]$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0.149565 \cdot 10^{-2}$</td>
<td>$0.151058 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>2</td>
<td>$0.200862 \cdot 10^{-2}$</td>
<td>$0.202867 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>3</td>
<td>$0.203892 \cdot 10^{-2}$</td>
<td>$0.205927 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>4</td>
<td>$0.200995 \cdot 10^{-2}$</td>
<td>$0.203001 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>5</td>
<td>$0.199978 \cdot 10^{-2}$</td>
<td>$0.201974 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>6</td>
<td>$0.199808 \cdot 10^{-2}$</td>
<td>$0.201802 \cdot 10^{-2}$</td>
</tr>
</tbody>
</table>

The polynomial response function between the Young modulus of this elastic beam and its deflection at the right hand side has been determined numerically using the polynomial interpolation option in the system MAPLE. This function has been shown in Fig. 9. and it was the basis for further symbolic derivation of higher order partial derivatives of this deflection with respect to the randomized Young modulus and this deflection probabilistic moments. The presentation of the computational results is quite similar to that given in Sec. 4.1 – probabilistic moments for the maximum deflection at the end of the cantilever beam are collected in Figs. 10-14: the expected values, the standard deviations, the variances as well as third and fourth order central probabilistic moments. The elastic foundation for the elastic beam having exactly the same parameters reduces the maximum deflection by two orders of the magnitude (by a comparison of Figs. 4 and 10), so that the standard deviations are reduced by two orders also (the following numbers are slightly different) the variances – by four orders (as the second powers of the standard deviations), whereas the third and fourth central probabilistic moments – by the six and eight orders, respectively. Finally, comparing the ascending order moments diagrams it is clear that the higher moment is computed the bigger influence of the perturbation parameter on the overall result is noticed.
Figure 9. The response function of the 6th node deflections of the beam

Figure 10. Expected values as the function of the perturbation parameter and input coefficient of variation (2nd, 4th and 6th orders)
Figure 11. Standard deviations as the function of the perturbation parameter and input coefficient of variation (2\textsuperscript{nd}, 4\textsuperscript{th} and 6\textsuperscript{th} orders)

Figure 12. Variances as the function of the perturbation parameter and input coefficient of variation (2\textsuperscript{nd}, 4\textsuperscript{th} and 6\textsuperscript{th} orders)
This study brings also the following important notice – looking for the variability of 3rd and 4th order probabilistic moments in the range of applicability of the second order second moment technique – within the interval [0.0, 0.1], cf. (Kleiber and Hien, 1992), one can observe that they are almost insensitive with respect to any parameter of this study and practically do not differ much from 0. Their largest increase is noticed for input coefficient of variation larger than 0.15 (the bigger coefficient – the larger moment increase).
5. CONCLUDING REMARKS

1. Theoretical considerations presented here and the additional computational implementation show how to make an efficient stochastic extension of any order for the well-known Finite Difference Method in the case, when some parameters in the governing differential equations remain uncertain. This uncertainty is represented by the truncated Gaussian input random variable defined uniquely by the first two probabilistic moments and, by the formation and straightforward solution of the ascending order equilibrium equations, results in probabilistic moments of the structural response. Although the entire methodology is displayed on the example of the elastic statistically homogeneous and isotropic beams, it remains so general that any 2 and/or 3D problems may be solved quite analogously. The probabilistic moments of up to the fourth order are computed symbolically, using the system MAPLE, so that, unlike most of the previous implementations (Kamiński, 2001, Kleiber and Hien, 1992), they include analytical functions with respect to the perturbation parameter as well as the input coefficient of variation for the random input. Furthermore, having in the mind the stochastic convergence of the entire method, we compare up to the tenth order perturbation theories in the case of the expected values and up to sixth order approaches in the case of the second order characteristics.

2. As one may expect, the computations performed shown that the coefficient of variation for the random input was determined as more important than the perturbation parameter. Further, the differences between the neighboring order approaches decrease together with the increase of this order, which partially demonstrates the convergence of this methodology. Comparison of the expected values for the beam without and with the elastic foundation shows that the application of this foundation decreases the maximum deflection by more than two orders of the magnitude. Finally, we need to point out that the Stochastic Finite Difference Method so proposed may be directly used for any order stochastic reliability studies since all the moments and probabilistic characteristics are easy available. Although the computer code is not very large, it can be generalized easily towards larger space dimensions of the structures, the other probability density functions as well as to automatic formation of the ascending order equations typical not only for the SFDM but also for the other discrete techniques implemented before.

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6. REFERENCES


