HOMOGENIZATION OF FIBER-REINFORCED COMPOSITES WITH RANDOM PROPERTIES USING THE LEAST-SQUARES RESPONSE FUNCTION APPROACH

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The main issue in this elaboration is computational study of the homogenized elasticity tensor for the periodic random composite using the improved stochastic generalized perturbation technique. The uncertainty of the composite appears at the component’s material properties, treated here as the Gaussian random variables, while its micro- and macrogeometry remains perfectly periodic. The effective modules method consisting in the cell problem solution is enriched with the generalized stochastic perturbation method. This method is implemented without the necessity of a large number of increasing order equations. The response function between the homogenized tensor and the input random parameter is determined numerically using several deterministic solutions and the least-squares approximation technique. Since classical polynomial approximation techniques may result in some errors for the lower and upper bound of the input parameter variability set, the least-squares approximation is used, where the degree of an approximant is the additional input variable. This approach has hybrid computational implementation—partially in the homogenization-oriented finite element method code MCCEFF and in the symbolic environment of the MAPLE 13 system, giving a wide range of approximation techniques that can also be modified in a graphical mode.

KEY WORDS: homogenization method, random composites, stochastic perturbation method, response function approach, least-squares method

1. INTRODUCTION

A consideration of the random periodic composite (Jeulin and Ostoja-Starzewski, 2001), where uncertainty appears in material parameters and geometry remains perfectly periodic, driven mainly by a number of engineering applications, as still a subject of many research works (Kamiński, 2009; Sakata et al., 2007; Tootkaboni and Graham-Brady, 2010). Statistical parameters for mechanical properties are determined using some laboratory experiments, but the measurement of statistical parameters for spatial distribution (location) of phases is not so easy and usually may be efficiently provided by the destructive testing (a posteriori and not on the design process level). Following this philosophy, we are going to compute the homogenized parameter statistics to be included in further reliability assessments. As is known (Nensoussan et al., 1978; Brandt, 1995; Cruz and Patera, 1995), it is possible to extend this model toward uncertainty in composite geometry using some numerical tessellation techniques (Ghosh et al., 1995), where both mechanical parameters and fiber locations may be treated as random.

A choice (or development) of the numerical method is adequate to the optimization of the final accuracy, time consumption, and overall computer power. Taking into account Monte Carlo simulation, we still develop the concurrent and decisively faster stochastic techniques. That is why the generalized stochastic perturbation method, having a priori given length of Taylor expansion [invented in Kamiński (2009) for a homogenization problem] and being fast enough, is still corrected and reimplemented. Now we obtain the smooth response functions between the effective elasticity tensor components and the input parameters within all variability of intervals thanks to application of the
least-squares approximation technique with a posteriori optimized approximation order. Its partially symbolic realization allows employment of new systematic mathematical approximation tools and verification of the overall accuracy in a graphical interactive way as presented below. It is seen that (Kamiński, 2005, 2009; Kamiński and Kleiber, 2000) the symbolic software application is the key numerical feature here, enabling polynomial expression of the output probabilistic moments, at least with respect to the stochastic perturbation parameter $\varepsilon$.

The key new feature in the computational apparatus used below is the least-squares approximation of the response function relating the homogenized elasticity tensor and elastic parameters of the composite constituents. This approximation is made on the basis of several deterministic solutions around the mean values of those parameters, and then the response functions are determined separately for all input material parameters of the fiber and matrix. An application of the least-squares nonlinear fitting appears to give a more reliable approximation of the response functions than the ordinary polynomial interpolation from the set of equidistant trial points employed in Kamiński (2009), but it is still convenient for the needs of the stochastic perturbation technique. (Lagrange polynomials are antiexample, since their further symbolic processing for higher order approximants results in significant computational times and power consumption.) Similar to the previous studies, now various order stochastic approximations of the homogenized tensor component expectations and standard deviations are tested but only with respect to the Poisson ratio of the matrix. This particular design parameter has been detected as the most influential parameter for the effective tensor components during the sensitivity analysis provided in Kamiński (2005, 2009).

2. HOMOGENIZATION METHOD

Let us introduce a geometrical scaling parameter $\zeta > 0$ between the micro- and macroscale of the composite (see Fig. 1) and introduce two coordinate systems $y = (y_1, y_2, y_3)$ on the microscale of the composite and $x = (x_1, x_2, x_3)$ on the macroscale. Let us denote the fiber region by $\Omega_1$, the matrix area by $\Omega_2$, and the interface between them as $\Gamma_{12}$ (continuous and smooth plane contour). Next, let us express any state function $G$ defined on $Y$ as

$$G^\zeta (x) = G \left( \frac{x}{\zeta} \right) = G (y).$$

(1)

The linear elasticity problem for the periodic composite structure is given as follows:

$$\begin{cases}
\frac{\partial \sigma^\zeta_{ij}}{\partial x_j} + F_i = 0 \\
\sigma^\zeta_{ij} n_j = p_i; \quad x \in \partial Y_o \\
u^\zeta_i = 0; \quad x \in \partial Y_u \\
\sigma^\zeta_{ij} = C^\zeta_{ijkl} (x) \, \varepsilon^\zeta_{kl} \\
\varepsilon^\zeta_{kl} = \frac{1}{2} \left( u^\zeta_{k,l} + u^\zeta_{l,k} \right)
\end{cases}$$

(2)

FIG. 1: Periodic composite structure $Y$
By assuming perfect interfaces between the matrix and the fibers, as well as no cracks and other defects into those constituents, we solve this problem by introducing the bilinear form \( a^\zeta(u, v) \),

\[
a^\zeta(u, v) = \int_\Omega C_{ijkl}(x) \varepsilon_{ij}(u) \varepsilon_{kl}(v) \, d\Omega,
\]

and the linear one

\[
L(v) = \int_\Omega F_i v_i d\Omega + \int_{\partial \Omega_p} p_i v_i (\partial \Omega) d\Omega,
\]

in the following Hilbert space of admissible displacements defined on \( Y \):

\[
V = \{ v \mid v \in [H^1(Y)]^3, \, v|_{\partial \Omega_u} = 0 \},
\]

\[
\|v\|^2 = \int_\Omega \varepsilon_{ij}(v) \varepsilon_{ij}(v) \, d\Omega.
\]

Then the variational statement equivalent to the equilibrium problem (2) is to find \( u^\zeta \in V \) being a solution of the following equation:

\[
a^\zeta(u^\zeta, v) = L(v), \quad v \in V.
\]

Let us define the additional space of the admissible displacement functions \( P(\Omega) = \{ v, \, v \in [H^1(\Omega)]^3 \} \) periodic on the composite cell \( \Omega \). We then introduce the new bilinear form for any \( u, v \in P(\Omega), \)

\[
a_y(u, v) = \int_\Omega C_{ijkl}(y) \varepsilon_{ij}(u) \varepsilon_{kl}(v) \, d\Omega,
\]

and the homogenization function \( \chi_{(ij)k} \in P(\Omega) \) (also of the displacement type) as a solution for the so-called local problem on a periodicity cell,

\[
a_y \left[ \chi_{(ij)k} + y_j \delta_{ki} \right] n_k, w = 0,
\]

for any \( w \in P(\Omega) \), where \( n_k \) is the unit coordinate vector. Assuming further boundedness, ellipticity, and symmetry of the fourth-order elasticity tensor one may perceive the effective elasticity tensor components from the following theorem (Bensoussan et al., 1978; Kamiński, 2005; Lené and Leguillon, 1982; Sanchez-Palencia, 1980):

The solution \( u^\zeta \) of problem (9) converges weakly in space \( V \),

\[
u^\zeta \rightarrow u \quad \text{as} \quad \zeta \rightarrow 0,
\]

for \( \Omega \)-periodic tensor \( C_{ijkl}^\zeta(y) \), where the solution \( u \) is the unique one for the problem

\[
u \in V, \quad S(u, v) = L(v)
\]

for any \( v \in V \) and

\[
S(u, v) = \int_Y S_{ijkl} \varepsilon_{ij}(u) \varepsilon_{kl}(v) dY,
\]

where

\[
S_{ijkl} = \frac{1}{[\xi]} a_y \left[ \chi_{(ij)p} + y_j \delta_{pi} \right] n_p, \left( \chi_{(kl)r} + y_i \delta_{rk} \right) n_r.
\]

Now we consider the boundary value problem in its differential form,

\[
\frac{\partial}{\partial x_j} \left[ C_{ijkl} \left( \frac{x}{\xi} \right) \varepsilon_{kl}(u^\zeta) \right] + F_i = 0; \quad u^\zeta_i = 0 \quad \text{for} \quad y \in \partial Y,
\]
and the following representation for the displacement using the parameter \( \zeta \) (Bensoussan et al., 1978; Lené and Leguillon, 1982; Sanchez-Palencia, 1980),

\[
u^\zeta(x) = \sum_{m=0}^{\infty} \zeta^m u^{(m)}(x, y),
\]

(15)

where any \( u^{(m)}(x, y) \) is periodic in \( y \) on \( \Omega \) and \( m = 0, 1, 2, \ldots \). The variables separation of \( x \) and \( y \) leads to the statement

\[
\varepsilon_{ij}(v) = \varepsilon_{ij}^x(v) + \frac{1}{\zeta} \varepsilon_{ij}^y(v),
\]

(16)

where, for instance,

\[
\varepsilon_{ij}^x(v) = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).
\]

(17)

Thus relation (14) is expanded as

\[
(\zeta^{-2}L_1 + \zeta^{-1}L_2 + L_3) \cdot \sum_{m=1}^{\infty} \zeta^m u^{(m)}(x, y) + F = 0,
\]

(18)

where

\[
L_1 u = \frac{\partial}{\partial y_i} [C_{ijkl}(y)] \varepsilon_{k\ell}^y(u),
\]

(19)

\[
L_2 u = C_{ijkl}(y) \frac{\partial}{\partial x_j} [\varepsilon_{k\ell}^y(u)] + \frac{\partial}{\partial y_i} [C_{ijkl}(y) \varepsilon_{k\ell}^x(u)],
\]

(20)

\[
L_3 u = C_{ijkl}(y) \frac{\partial}{\partial x_j} [\varepsilon_{k\ell}^x(u)].
\]

(21)

Next, equating the terms with the same order of \( \zeta \) to 0, the following equations of zeroth, first, and second order are obtained:

\[
L_1 u^{(0)} = 0,
\]

(22)

\[
L_1 u^{(1)} + L_2 u^{(0)} = 0,
\]

(23)

\[
L_1 u^{(2)} + L_2 u^{(1)} + L_3 u^{(0)} + F = 0,
\]

(24)

from which the functions \( u^{(0)}, u^{(1)}, \) and \( u^{(2)} \) are determined. Let us also note that the equation

\[
L_1 u + P = 0
\]

(25)

for \( u \) being a \( \Omega \)-periodic function has a unique solution for

\[
\langle P \rangle = \frac{1}{|\Omega|} \int_{\Omega} P \, dy = 0.
\]

(26)

Starting from the above equations, it yields

\[
u^{(0)}(x, y) = u(x),
\]

(27)

while Eq. (23) takes the following form:

\[
L_1 u^{(1)}(x, y) + \frac{\partial}{\partial y_i} [C_{ijkl}(y)] \varepsilon_{k\ell}^x \left[ u^{(0)}(x) \right] = 0.
\]

(28)

The separation process of the variables \( x \) and \( y \) leads to

\[
u^{(1)}_i(x, y) = X_{(kl)i}(y) \varepsilon_{k\ell}^x \left[ u^{(0)}(x) \right] + u_i(x).
\]

(29)
The last two equations give the formulation for the $\Omega$-periodic functions $\chi_{(kl)i}(y)$ as

$$\frac{\partial}{\partial y_i} \left( C_{ijkl}(y) \frac{\partial \chi_{(kl)m}(y)}{\partial y_m} \right) + \frac{\partial}{\partial y_i} C_{ijkl}(y) = 0, \quad (30)$$

so that the local problem of the homogenization function $\chi_{(kl)i}(y)$ reduces to the equations posed above and the following conditions hold true:

$$\chi_{(kl)i} \text{ periodic on } \Omega, \quad (31)$$

and

$$\sigma_{ij} \left( \chi_{(pq)i} \right) n_j = [C_{pqij}]_{1_{\Gamma_{12}}} \quad \text{on } \Gamma_{12}. \quad (32)$$

Hence the variational formulation necessary for the displacement version of the finite-element method (FEM) analysis of the local problem can be written out as

$$C_{ijkl}^{(1)} \int_{\Omega_1} \varepsilon_{kl} \left( \chi_{(pq)i} \right) \varepsilon_{ij} (v) d\Omega + C_{ijkl}^{(2)} \int_{\Omega_2} \varepsilon_{kl} \left( \chi_{(pq)i} \right) \varepsilon_{ij} (v) d\Omega = - \int_{\Gamma_{12}} \sigma_{ij} \left( \chi_{(pq)i} \right) n_j v_i d\Gamma. \quad (33)$$

After solving for $\chi_{(pq)i}$, from Eq. (33) one computes the effective elasticity tensor as

$$C_{ijkl}^{(eff)} = \frac{1}{|\Omega|} \int_{\Omega} \left[ C_{ijpq} + C_{ijkl} \varepsilon_{pq} \left( \chi_{(kl)i} \right) \right] d\Omega. \quad (34)$$

Furthermore, it is clear that if the second component of the right-hand side integrand function is omitted, well-known upper bounds for the effective elasticity tensor for the composite are returned (Christensen, 1979; Milton, 2002). The experimental and computational analyses prove that these bounds are easy to calculate, even for the case of random spaces of composite constituent elastic characteristics, but their values differ significantly from the real effective properties. Let us finally note that since the probabilistic method uses several deterministic solutions to complete the entire homogenization of a random composite, this mathematical apparatus does not need further improvement and the homogenization theorem remains valid. Additionally, we assume only material characteristics of the constituents to be truncated and independent Gaussian random variables (limited to the physically admissible intervals) having uniquely defined a priori the first two probabilistic moments.

3. GENERALIZED STOCHASTIC PERTURBATION TECHNIQUE IN HOMOGENIZATION

The probabilistic perturbation methodology is proposed to calculate the moments of the homogenized elasticity tensor discussed above. It is based on an expansion via the Taylor series regarding the spatial expectations using a small parameter $\varepsilon > 0$ and for some random output, the following expression is employed:

$$C_{ijkl}^{(eff)} (b) = C_{ijkl}^{(eff)0} + \varepsilon \frac{\partial C_{ijkl}^{(eff)}}{\partial b} \Delta b + \frac{1}{2} \varepsilon^2 \frac{\partial^2 C_{ijkl}^{(eff)}}{\partial b^2} \Delta b \Delta b + \ldots + \frac{1}{n!} \varepsilon^n \frac{\partial^n C_{ijkl}^{(eff)}}{\partial b^n} (\Delta b)^n, \quad (35)$$

where $b$ stands for the random input and $\Delta b$ denotes the variation of this variable around its expected value; the accuracy of this expansion depends heavily on the perturbation order. The recursive formula for the central $m$th-order probabilistic moment in tenth-order approximation can be determined as Bendat and Piersol (1971)

$$\mu_m \left[ C_{ijkl}^{(eff)} (b) \right] = \int_{-\infty}^{+\infty} \left( C_{ijkl}^{(eff)0} (b) + \sum_{i=1}^{n} \frac{\varepsilon^i}{i!} \frac{\partial^i C_{ijkl}^{(eff)} (b)}{\partial b^i} \right) \ p(b) \ db$$

$$= \int_{-\infty}^{+\infty} \left( C_{ijkl}^{(eff)0} (b) + \varepsilon \frac{\partial C_{ijkl}^{(eff)} (b)}{\partial b} \Delta b + \frac{\varepsilon^2}{2!} \frac{\partial^2 C_{ijkl}^{(eff)} (b)}{\partial b^2} \Delta b \Delta b \right) \ p(b) \ db$$

$$= \int_{-\infty}^{+\infty} \left( \varepsilon \frac{\partial C_{ijkl}^{(eff)} (b)}{\partial b} \Delta b + \frac{\varepsilon^2}{2!} \frac{\partial^2 C_{ijkl}^{(eff)} (b)}{\partial b^2} (\Delta b)^2 \right) \ p(b) \ db. \quad (36)$$
Following this expression, the expected value and the variance of the resulting homogenized tensor $C^{(eff)}_{ijkl}(b)$ can be obtained for $b$ Gaussian:

$$
E \left[ C^{(eff)}_{ijkl} \right] = C^{(eff)}_{ijkl}(b) + \frac{1}{2} \frac{\partial^2 C^{(eff)}_{ijkl}(b)}{\partial b^2} \times \text{Var}(b)
$$

$$
+ \frac{1}{4!} \times \frac{\partial^4 C^{(eff)}_{ijkl}(b)}{\partial b^4} \times \mu_4 + \frac{1}{6!} \times \frac{\partial^6 C^{(eff)}_{ijkl}(b)}{\partial b^6} \times \mu_6 + \ldots + \frac{1}{2m!} \times \frac{\partial^{2m} C^{(eff)}_{ijkl}(b)}{\partial b^{2m}} \times \mu_{2m}(b)
$$

(37)

and furthermore, assuming $C^{(eff)}_{ijkl}(b) = E \left[ C^{(eff)}_{ijkl}(b) \right]$, we have:

$$
\text{Var} \left( C^{(eff)}_{ijkl} \right) = \text{Var}(b) \times \left( \frac{1}{3!} \right)^2 \frac{\partial^3 C^{(eff)}_{ijkl}}{\partial b^3} + 1 \frac{4}{4!} \frac{\partial^4 C^{(eff)}_{ijkl}}{\partial b^4} \frac{\partial^2 C^{(eff)}_{ijkl}}{\partial b^2} \frac{\partial C^{(eff)}_{ijkl}}{\partial b} + 2 \frac{5}{5!} \frac{\partial^5 C^{(eff)}_{ijkl}}{\partial b^5} \frac{\partial^3 C^{(eff)}_{ijkl}}{\partial b^3} \frac{\partial C^{(eff)}_{ijkl}}{\partial b} + \ldots
$$

(38)

where $\mu_k(b)$ is the $k$th-order central probabilistic moment of variable $b$. The perturbation parameter is adopted as $\varepsilon = 1$ and only the first few perturbations are included, especially in the last relation. Quite similarly, using the first- and second-order terms only, it is possible to derive third-order probabilistic moments as

$$
\mu_3 \left[ C^{(eff)}_{ijkl}(b) \right] = \int_{-\infty}^{+\infty} C^{(eff)}_{ijkl}(b) \frac{\partial C^{(eff)}_{ijkl}}{\partial b} \Delta b + \frac{1}{2} \varepsilon^2 \frac{\partial^2 C^{(eff)}_{ijkl}}{\partial b^2} \Delta b \Delta b + \ldots - E \left[ C^{(eff)}_{ijkl}(b) \right]^3 p(b) db
$$

(39)

and the fourth-order probabilistic moment also,

$$
\mu_4 \left[ C^{(eff)}_{ijkl}(b) \right] = \int_{-\infty}^{+\infty} C^{(eff)}_{ijkl}(b) \frac{\partial C^{(eff)}_{ijkl}}{\partial b} \Delta b + \frac{1}{2} \varepsilon^2 \frac{\partial^2 C^{(eff)}_{ijkl}}{\partial b^2} \Delta b \Delta b + \ldots - E \left[ C^{(eff)}_{ijkl}(b) \right]^4 p(b) db
$$

(40)

In all those equations no summation on $i, j, k, l$ takes place. One also defines the coefficients of variation, asymmetry, and concentration as

$$
\alpha \left[ C^{(eff)}_{ijkl}(b) \right] = \sqrt{\frac{\text{Var}(C^{(eff)}_{ijkl})}{E(C^{(eff)}_{ijkl})}}; \quad \beta \left[ C^{(eff)}_{ijkl}(b) \right] = \frac{\mu_3 \left[ C^{(eff)}_{ijkl}(b) \right]}{E(C^{(eff)}_{ijkl})} \sqrt{\text{Var}(C^{(eff)}_{ijkl})}; \quad \gamma \left[ C^{(eff)}_{ijkl}(b) \right] = \frac{\mu_4 \left[ C^{(eff)}_{ijkl}(b) \right]}{\left[ \text{Var}(C^{(eff)}_{ijkl}) \right]^2}.
$$

(41)

As it can be seen in these equations, the symbolic approach perfectly reflects the needs of higher order perturbation approaches, where the perturbation parameter $\varepsilon$ with increasing powers can also be inserted directly in the Taylor series expansion. This opportunity is realized in numerical experiments to compute symbolically up to the tenth-order expression for interphase homogenized parameters to eliminate the deficiencies of the perturbation technique itself. It should be clearly noted that those equations are still independent of the probability distribution type. The Gaussian and

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lognormal distributions are decisively preferred here, since recursive formulas for all probabilistic moments may be easily implemented into the symbolic computer programs; otherwise some extra mathematical transforms or a priori assumptions need to be done. Only the distributions having the analytical function generating probabilistic moments are applicable here.

4. COMPUTATIONAL IMPLEMENTATION

Let us introduce the following approximation of homogenization functions $\chi_{(uv)i}^\delta$ at any point of the considered continuum $\Omega$ in terms of a finite number of generalized coordinates $q_{(uv)\alpha}$ and shape functions $\varphi_{i\alpha}$ (Kamiński, 2005, 2009; Kamiński and Kleiber, 2000):

$$\chi_{(uv)i}^\delta = \varphi_{i\alpha} q_{(uv)\alpha}, \quad i, u, v = 1, 2, \quad \alpha = 1, \ldots, N, \delta = 1, \ldots, L. \quad (42)$$

The variables $i, u, v$ stand for the spatial coordinates, $\alpha$ denotes the degree of freedom, where $N$ is the total number of degrees of freedom, and $\delta$ denotes the current least-squares method test number, where $L$ is the a priori chosen total number of those tests. We discretize analogously the strain $\varepsilon_{ij} \left( \chi_{(uv)k}^\delta \right)$ as well as stress tensors $\sigma_{ij} \left( \chi_{(uv)k}^\delta \right)$:

$$\varepsilon_{ij} \left( \chi_{(uv)}^\delta \right) = B_{ijkl} q_{(uv)\delta}, \quad (43)$$

$$\sigma_{ij}^\delta = \sigma_{ij} \left( \chi_{(uv)}^\delta \right) = C_{ijkl}^\delta \varepsilon_{kl} \left( \chi_{(uv)}^\delta \right) = C_{ijkl}^\delta B_{kl\alpha} q_{(uv)\alpha}, \quad (44)$$

where $B_{kl\alpha}$ is the shape function derivatives matrix, conventional in the FEM, which does not vary on the least-squares approximation test. Therefore the virtual work equation looks like

$$\int_{\Omega} \delta \chi_{(uv)i,j}^\delta C_{ijkl}^\delta \chi_{(uv)k,l}^\delta d\Omega = - \int_{\Gamma_{12}} \delta \chi_{(uv)i}^\delta \left[ F_{(uv)i}^\delta \right] |_{\Gamma_{12}} d\Gamma \quad \text{(no summation on u, v)}. \quad (45)$$

The FEM formulation continues with a definition of the global stiffness matrix, which proceeds as follows:

$$K_{\alpha\beta}^\delta = \sum_{e=1}^{E} K_{\alpha\beta}^{(e)\delta} = \sum_{e=1}^{E} \sum_{l=1}^{L} C_{ijkl}^{(e)\delta} B_{ijkl} B_{kl\alpha} d\Omega. \quad (46)$$

Introduction of this matrix into Eq. (45) and minimization with respect to the generalized coordinates enables

$$K_{\alpha\beta} q_{(uv)\beta}^\delta = Q_{\alpha\beta}^\delta q_{(uv)\alpha}^\delta, \quad (47)$$

where the right-hand vector consists of the stress interface conditions also varying on the least-squares test. The symmetry conditions on the periodicity cell quarter are assumed, so that the orthogonal displacements for every nodal point belonging to the external boundaries of $\Omega$ are fixed to compute $L$ homogenization functions ($\chi_{(uv)i}^\delta$ for $\delta = 1 \ldots L$) and the resulting homogenizing stress fields. Finally, they are spatially averaged into all finite elements constituting the representative volume element (RVE) and combined with the original additional elasticity tensor components according to Eq. (34):

$$C_{ijkl}^{(eff)\delta} = \frac{1}{|\Omega|} \int_{\Omega} \left[ C_{ijkl}^\delta + C_{ijkl}^\delta \varepsilon_{pq} \chi_{(kl)q}^\delta \right] d\Omega. \quad (48)$$

Furthermore, we use the following polynomial representation of the homogenized elasticity tensor and its $s$th-order partial derivatives as

$$C_{ijkl}^{(eff)} = \sum_{r=1}^{n} A_{ijkl}^{(r)} b^r, \quad (49)$$
So the expected values are extracted here as

\[ P_r^{(r)} P(r - s + 1, s) b^{r - s}, \quad p \leq s, \]

\[ 0, \quad p > s \]  

as long as \( n \leq \delta \), where the coefficients \( A_{ijkl}^{(r)} \) are found using the least-squares approximation procedure and where \( P(r - s + 1, s) \) denotes the Porchhammer symbol, usually defined as

\[ P_z(z, n) = z(z + 1)(z + 2)\ldots(z + n - 1) = \frac{\Gamma(z + n)}{\Gamma(z)}. \]  

(51)

So the expected values are extracted here as

\[
E \left[ C_{ijkl}^{(eff)} \right] = C_{ijkl}^{(eff)0} + \frac{1}{2!} \varepsilon^2 \mu_2 (b) \sum_{p=1}^{n} p(p-1) A_{ijkl}^{(p)} b^{p-2} \\
+ \frac{1}{4!} \varepsilon^4 \mu_4 (b) \sum_{p=1}^{n} p(p-1)(p-2)(p-3) A_{ijkl}^{(p)} b^{p-4} + \frac{1}{6!} \varepsilon^6 \mu_6 (b) \sum_{p=1}^{n} p(p-1)(p-2)(p-3) A_{ijkl}^{(p)} b^{p-6} \\
+ \frac{1}{8!} \varepsilon^8 \mu_8 (b) \sum_{p=1}^{n} p(p-1)(p-2)(p-3)(p-4) A_{ijkl}^{(p)} b^{p-8} + \frac{1}{10!} \varepsilon^{10} \mu_{10} (b) \sum_{p=1}^{n} p(p-1)(p-2)(p-3)(p-4) A_{ijkl}^{(p)} b^{p-10}. 
\]

(52)

Higher order moment determination proceeds similarly. Finally, we notice that for the randomized material parameters we can apply a semianalytical approach, where integrand function is represented using the classical definition as

\[
C_{ijkl}^{(eff)} = \left[ \delta_{ij} \delta_{pq} \frac{E \nu}{(1 + \nu)(1 - 2\nu)} + (\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp}) \frac{E\nu}{2(1 + \nu)} \right]_{\Omega} + \left[ C_{ijkl} \varepsilon_{pq} \left( \chi_{(kl)} \right) \right]_{\Omega}. 
\]

(53)

The partial derivatives of the first component with respect to the given parameter \( b \) are determined straightforwardly, and the second component needs some least-squares method approximants.

5. PROBABILISTIC COMPUTER ANALYSIS OF THE HOMOGENIZED ELASTICITY TENSOR

A numerical analysis of the periodic random fiber composite homogenization is performed using the specially adopted FEM and homogenization-oriented program MCCEFF (Kamiński, 2005) and the symbolic computing environment of the MAPLE system (ver. 13). The internal automatic generator of this program for the square RVE with a centrally located round fiber occupying 34% of the RVE is used to prepare the mesh consisting of 144 four-noded plane-strain finite elements and 153 nodes (Kamiński, 2005, 2009; Kamiński and Kleiber, 2000). The elastic parameters of the fiber material are taken as \( E_1 = 84 \text{ GPa}, \nu_1 = 0.22 \), and for the matrix \( E_2 = 4 \text{ GPa}, \nu_2 = 0.34 \) (its expected value). According to the previous studies in this area, the Poisson ratio for the matrix has been detected for this composite as \( \nu_2 = 0.34 \) (its expected value). The additional input parameter of this analysis, which is given each time on the horizontal axis. The expected values and standard deviations for all the homogenized tensor components are computed for the first few perturbation orders to verify the probabilistic convergence of this method (Kamiński, 2009), shows no local variations at the trial interval ends.

Now we randomize this Poisson coefficient of the matrix using the coefficient of variation \( \alpha \) as the additional input parameter of this analysis, which is given each time on the horizontal axis. The expected values and standard deviations for all the homogenized tensor components are computed for the first few perturbation orders to verify the probabilistic convergence of this method (Kamiński, 2009), shows no local variations at the trial interval ends.

\[ C_{ijpq}^{(eff)} = \left[ \delta_{ij} \delta_{pq} \frac{E \nu}{(1 + \nu)(1 - 2\nu)} + (\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp}) \frac{E\nu}{2(1 + \nu)} \right]_{\Omega} + \left[ C_{ijkl} \varepsilon_{pq} \left( \chi_{(kl)} \right) \right]_{\Omega}. 
\]

(53)
asymmetry and concentration of the single homogenized tensor component to prove the applicability of the proposed method for also computing higher probabilistic moments.

As it is clear from Figs. 3–5, usually the second-order approach is acceptable for the very small input coefficient of variation (according to the previous predictions), but for $\alpha > 0.10$ higher order terms really need to be included. Higher order analysis leads immediately to the conclusion that for $\alpha < 0.20$, the tenth-order approach has sufficient accuracy for the expected values of all components for the homogenized tensor. A contrast of Figs. 4 and 5 shows that the tenth-order analysis does not always result in the largest magnitudes of the expectations—sometimes a probabilistic convergence has asymptotic character; the differences between the neighboring order approximations systematically decrease anyway. As one could expect after deterministic sensitivity analysis, the largest differences are noticed in Fig. 4, because this particular component demonstrates the largest sensitivity coefficients with respect to $\nu_2$. Those coefficients are also computed in this approach and can be extracted from the first-order partial derivatives of $C_{ijkl}^{(eff)}$. Let us note here that contrary to the statistical estimation methods, the expected values demonstrate little variability with respect to the input coefficient of variation, which is the inherent aspect of the entire stochastic perturbation technique.

It is obvious that the standard deviations show significantly slower probabilistic convergence and that they have parameter variability significantly closer to the linear function. Now even for $\alpha > 0.10$ the differences between the
lowest orders of those deviations are apparent and should not be neglected. Contrary to the expectations, now all new orders systematically increase the final approximation results. Let us remember that real engineering materials do not exhibit such large standard deviations as 0.2; however, considering other input parameters it shows a possible range of homogenized tensor random fluctuations.
Next we examine in Figs. 9 and 10 the coefficients of asymmetry and concentration, also parameterized with the perturbation parameter $\varepsilon = 0.9 \ldots 1.1$ as it is presented analytically in Eqs. (39) and (40). It confirms that the fourth-order characteristic is almost entirely influenced by this parameter choice and is completely insensitive to the input coefficient of variation. Here computer analysis gives $\gamma = 3$ for $\varepsilon = 1$, which is typical for the Gaussian distribution.
FIG. 8: Various order standard deviations of $C^{(\text{eff})}_{1212}, \nu_2 = \nu_2(\omega)$

FIG. 9: Coefficients of asymmetry for $C^{(\text{eff})}_{1212}, \nu_2 = \nu_2(\omega)$

FIG. 10: Coefficients of concentration for $C^{(\text{eff})}_{1212}, \nu_2 = \nu_2(\omega)$
The third-order coefficient is dominated by \( \alpha \) and is less influenced by \( \varepsilon \), however for all combinations appears to be positive and nowhere equal to zero as for the Gaussian variables. The results obtained for the generalized stochastic perturbation technique are contrasted in Fig. 10 with the Monte Carlo simulation results obtained for \( 10^4 \) random trials marked here with an asterisk. All the simulations have been provided for \( \alpha = 0.0, 0.01, \ldots, 0.15 \) and \( \varepsilon = 1 \) and, as is apparent, the simulation results are slightly larger each time than the stochastic perturbation technique results. Thus the output probability density functions are recognized, thanks to both methods, as very close to the Gaussian distributions, which remains in perfect agreement with the previous Monte Carlo simulations (Kamiński, 2005, 2009).

6. CONCLUDING REMARKS

1. The least-squares response function–based homogenization method shows faster probabilistic convergence than the traditional polynomial response approach presented before (Kamiński, 2009). It also enables for more accurate computation of higher probabilistic moments of \( C_{ijkl}^{(eff)} \), which have values very similar to those estimated by the corresponding Monte Carlo simulation method (see also (Kamiński, 2005; Kamiński and Kleiber, 2000) for other applications). One can also consider a replacement of the global least-squares approximation for the homogenizing stresses with the local version, where the response functions are determined separately for each node within the periodicity cell mesh.

2. The consistency of probabilistic moments computed here with the Monte Carlo simulation modeling and relatively short time of computations allows for more advanced constitutive models to be implemented into the homogenization procedure [including thermal stresses or temperature-dependent properties (Golansky et al., 1997)], especially for the elastoplastic constituents. On the other hand, the method needs to be extended toward the homogenization of composites with random microstructures also taken from the digital image processing (Terada et al., 1997). However, the question remains regarding the convergence of the expansion expressed by Eqs. (22)–(24) [higher order approximation (Fish and Chen, 2001)], where only the first term is included in the homogenized tensor formula.

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