Homogenization of metallic fiber-reinforced composites under stochastic ageing

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Abstract

The main aim of this paper is to present an application of the generalized stochastic perturbation technique to model stochastic ageing processes of the metallic fibre-reinforced periodic composite materials in terms of their effective properties. Those ageing processes are modelled here as two-parametric time series having Gaussian random initial values and time rate, both defined uniquely by their expectations and standard deviations. Computational homogenization procedure is discrete and based on the Finite Element Method program MCCEFF as well as the computer algebra system MAPLE, where the Response Function Method and the stochastic analysis are entirely implemented. This numerical strategy is used to analyze probabilistic moments of the effective elastic tensor of the few metal matrix composites as well as to simulate stochastic ageing of two representative composites – MoSio2–SiC and Ti–SiC. The approach proposed and results of computations may be further applied in the reliability analysis of metallic or the other composites.

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1. Introduction

A computational modeling of the stochastic processes based on the Monte-Carlo simulation method is known for their large time consumption because the entire generalizations of random populations in different discrete time moments are evaluated and used in various Finite Element Method models. More optimal strategy would be based on the observation of the probabilistic moments of the examined processes in the same time moments as before performed with at least comparable accuracy. The generalized stochastic perturbation technique based on the Taylor expansion of the uncertain parameters and the state functions may be useful in this case, to determine for instance the probabilistic moments of the effective elasticity tensor. However, we need to have the analytical description for the basic moments of the stochastic processes being modeled to assure the sufficient input for the perturbation-based analysis. This process does not need to have the Gaussian realizations for the whole history, however the lack of correlation between various parameters simplifies significantly the analysis. We can use in this purpose some forms of the ageing processes known from computational biology [1] (as the exponential forms), power-laws popular in various branches of engineering [2] or just the linear decay [4] with random coefficients popular in the civil engineering inspections.

Contrary to the second order second moment implementations of the perturbation technique, the hierarchical equations are not solved here for the increasing order approximants for the probabilistic output. Now, the Response Function Method (RSM) is explored, where the polynomial interrelation between the stochastic output and input is to be approximated symbolically via several deterministic solutions around the mean values of the stochastic parameters in various time moments of the process. Finally, one can obtain a discrete polynomial approximation of the stochastic process as the function of the initial stochastic process of the ageing, for instance. The method is similar to the Response Surface Method known from the literature [9], but instead of the polynomial form of the lower order for multiple parameters we use here higher order approximation for a single variable. It should be underlined that the use of multiple variables is also allowable, however we would like to distinguish between the influence of different physical quantities influencing the time fluctuations of the effective tensor probabilistic characteristics. We need to emphasize that the classical Finite Element Method [11] programs (with and without the access to the solver source code) may be employed and extended using the proposed approach.

The engineering practice with composites (and even classical homogeneous engineering materials) shows that the ageing of materials (neglecting the real nature of this mechanism) is dangerous for many structures and elements and should be included into the designing process. An integral part of such a designing process should be a mathematical equation simulating the ageing behavior of various structural elements and materials, a proper computational modeling technique as well as the final conclusion stating the safe time of operation for the specific engineering structure in the given environment. The mathematical equation responsible for the ageing process may be proposed using the strength verification, however it would need the very large number of...
experimentation extended for the very long periods of time. Instead of it, one may adopt some ageing model proposed extensively in the literature and following the research made in various applied science and engineering branches – like exponential or linear ageing decay for different physical or mechanical parameters.

Taking into account the above considerations, the stochastic ageing laws applied for the Young moduli of the composite components are examined in terms of the effective parameters for the periodic fiber-reinforced composites with metallic components. The linear decay responsible for the ageing process is adopted, while the initial value and the ageing velocity are the Gaussian random parameters having first two moments constant in time. This definition of the stochastic process enables for the ageing process to be modeled, where the response functions are found and used to finally determine stochastic processes and where graphical representations for the stochastic processes are provided. Let us underline that the comparison of the tenth order perturbation approach with the Monte-Carlo simulation results were proven before in various computational experiments with random variables [3–5], so that it is employed here to discuss the influence of probabilistic and stochastic fluctuations in Young moduli of metallic composites on the probabilistic moments for their composite effective parameters.

2. Fibre-reinforced composite model

The periodic fiber-reinforced composite structure in the plane strain with linear elastic and transversely isotropic components and some stochastic parameters is considered now. Let us denote the Representative Volume Element (RVE) of this composite as $\Omega$: $\mathbb{R}^2$ stands for the section of this composite with $x_3 = 0$ plane being constant along the $x_3$ axis parallel to the fibers direction (Fig. 1).

Let the region $\Omega$ contain two perfectly bonded, coherent and disjoint subsets $\Omega_1$ (fiber) and $\Omega_2$ (matrix) and let the scale between corresponding geometrical diameters of $\Omega$ and $Y$ is described by some small and real parameter $\varepsilon > 0$. Let $\partial \Omega$ denote external boundary of the $\Omega$ while $\partial \Omega_{\Omega_2}$ – the interface boundary between $\Omega_1$ and $\Omega_2$ regions. The elasticity tensor is defined here as

$$C_{ijkl}(x) = B_{ijkl}(x)\varepsilon(x)$$

$$= \varepsilon(x)\left\{\begin{array}{l}
\delta_{ij}\delta_{kl} \frac{1}{1-v(x)}(1-2v(x)) \\
(\delta_{ik}\delta_{jl} + \delta_{ij}\delta_{kl}) \frac{2}{1-v(x)}
\end{array}\right\}$$

The effective tensor $C_{ijkl}^{(eff)}$ (for the artificial homogenized composite) is introduced as such a tensor that replacing $C_{ijkl}$ (for the real composite) with $C_{ijkl}^{(eff)}$ in the following equilibrium equations:

$$C_{ijkl}^{(eff)}(x') + f_i = 0, \ x \in \Omega$$

$$\varepsilon_{ij}(x') = \frac{1}{2}(u_{ij}' + u_{ji}'), \ x \in \Omega$$

$$C_{ijkl}(x') = \chi_1(x')(C_{ijkl}^{(1)} + (1 - \chi_1(x'))C_{ijkl}^{(2)})$$

where $u'$ is obtained as a solution being a weak limit of $u'$ with $\varepsilon$ → 0 and where the characteristic function defining the elastic parameters equals to

$$\chi_1(x) = \begin{cases} 1, & x \in \Omega_1 \\ 0, & x \in \Omega_2 \end{cases}$$

with the boundary conditions

$$u' = 0, \ x \in \partial \Omega.$$

Now, let us consider the stochastic variations in Young modulus within the composite, i.e. [4]

$$C_{ijkl}(x; \omega, t) = B_{ijkl}(x)(\varepsilon(x; \omega, t),$$

with the following representation:

$$\varepsilon(x; \omega, t) = -\varepsilon(x; \omega,t) + \varepsilon_0(x; \omega).$$

The random field $\varepsilon_0(x; \omega)$ is equivalent to the initial Young moduli of the composite constituents, whereas $\varepsilon(x; \omega)$ represents the velocity of ageing process for the matrix and the fibre separately, i.e.

$$\varepsilon(x; \omega) = \chi_1(x)\varepsilon_1(\omega) + (1 - \chi_1(x))\varepsilon_2(\omega).$$

It is assumed that this process is of course continuous in time and the particular components are fully uncorrelated from each other (Young modulus of both components). Taking into account the relation (5) one can write that

$$E[\varepsilon(x; \omega)] = \chi_1 E[\varepsilon_1] + (1 - \chi_1)E[\varepsilon_2]$$

for $\chi_1 = \begin{cases} 1, & x \in \Omega_1 \\ 0, & elsewhere \end{cases}$

the variance is defined accordingly as

$$Var(\varepsilon(x; \omega)) = \chi_1 Var(\varepsilon_1) + (1 - \chi_1)Var(\varepsilon_2).$$

3. Homogenization method

Problem: Determine the series of probabilistic moments $\mu_n\left(C_{ijkl}^{(eff)}\right)$ for $\Omega$ using the lemma

$$C_{ijkl}^{(eff)} = \frac{1}{|\Omega|} \int_{\Omega} (C_{ijkl}(y) + C_{ijkl}(y)\varepsilon_0(x; \omega))\ v(x) \ v(\omega)$$

with periodic and kinematically admissible homogenization function $\chi_{ijkl}$ being a solution to the local problem on $Y$:

$$a_\varepsilon'(a_\varepsilon, \omega) = 0$$

for any periodic $w$ ($n_k$ is the unit coordinate vector). A bilinear form

$$a'(u, v) = \int_{\Omega} C_{ijkl}(x)\varepsilon_{ij}(u)\varepsilon_{kl}(v)\ v(x)$$

together with the linear one (including body forces and von Neumann boundary conditions)

$$L(v) = \int_{\Omega} f_i\ v_i\ d\Omega + \int_{\partial\Omega} p_i\ v_i\ d\partial\Omega.$$

![Fig. 1. Periodic fiber reinforced composite.](image-url)
lead via comparison in the variational statement of the local problem for \( \mathbf{u}^r \)

\[
\sigma^r(\mathbf{u}^r, \mathbf{v}) = L(\mathbf{v}), \quad \mathbf{v} \in V
\]  

(16)

where the scale transformation is given by

\[
u(\mathbf{X}) = \mathbf{u}(\frac{\mathbf{X}}{\varepsilon}) = \mathbf{u}(\mathbf{y}).
\]  

(17)

The Hilbert space of admissible displacements periodic on \( Y \) is defined as

\[
V = \{ \mathbf{v} | \mathbf{v} \in (H^1(Y))^3, \mathbf{v}_{|_{\partial Y_0}} = 0 \},
\]  

(18)

\[
\| \mathbf{v} \|^2 = \int_Y \varepsilon_l(\mathbf{v}) \varepsilon_l(\mathbf{v}) d\Omega.
\]  

(19)

Since with \( \varepsilon \to 0 \) \( \mathbf{u}^r \) converges weakly

\[
\mathbf{u}^r \to \mathbf{u}, \quad \mathbf{u} \in V : D(\mathbf{u}, \mathbf{v}) = L(\mathbf{v})
\]  

(20)

for any admissible displacement \( \mathbf{v} \) and

\[
\begin{aligned}
D(\mathbf{u}, \mathbf{v}) &= \int_{\partial D} \varepsilon_l(\mathbf{u}_l) \varepsilon_l(\mathbf{v}_l) d\Omega + \int_{\partial D} \varepsilon_l(\mathbf{u}_l) \varepsilon_l(\mathbf{v}_l) d\Omega \\
D \mathbf{u} &= \int_{\partial D} \varepsilon_l(\mathbf{u}_l) \varepsilon_l(\mathbf{v}_l) d\Omega.
\end{aligned}
\]  

(21)

Computational determination of the homogenization function \( \chi_{\Omega}(Y) \) proceeds directly by dropping off the body forces as

\[
C^{(1)}_{ijkl} \int \varepsilon_l(\chi^{(p)}_{ijkl}) \varepsilon_l(t) d\Omega + C^{(2)}_{ijkl} \int \varepsilon_l(\chi^{(p)}_{ijkl}) \varepsilon_l(t) d\Omega
\]  

\[= - \int_{Y_{12}} \sigma_l(\chi^{(p)}_{ijkl}) n_l d\Gamma.
\]  

(22)

and with the following interface stress condition:

\[
\sigma_l(\chi^{(p)}_{ijkl}) n_l = [\mathbf{n}_{ijkl}]_{Y_{12}} n_l = \mathbf{F}_{ijkl} n_l = C^{(p)}_{ijkl} - C^{(1)}_{ijkl}
\]  

(23)

4. Stochastic analysis

The physical and engineering applications are usually focused on the expectations, variances, standard deviations, coefficients of variation, third and fourth probabilistic moments as well as on skewness and kurtosis for the structural random response. So that we introduce the additional integral definitions of probabilistic moments for each component of \( C^{(p)}_{ijkl} \) [3,7,8]:

- expected value

\[
E[C_{ijkl}^{(p)}] = \int_{-\infty}^{\infty} C_{ijkl}^{(p)} p(C_{ijkl}^{(p)}) d\mathbf{x},
\]  

(24)

- variance

\[
\text{Var}[C_{ijkl}^{(p)}] = \int_{-\infty}^{\infty} \left( C_{ijkl}^{(p)} - E[C_{ijkl}^{(p)}] \right)^2 p(C_{ijkl}^{(p)}) d\mathbf{x},
\]  

(25)

- standard deviation and coefficient of variation

\[
\sigma(C_{ijkl}^{(p)}) = \sqrt{\text{Var}[C_{ijkl}^{(p)}]},
\]  

\[
\alpha(C_{ijkl}^{(p)}) = \frac{\sqrt{\text{Var}[C_{ijkl}^{(p)}]}}{E[C_{ijkl}^{(p)}]},
\]  

(26)

- \( k \)th central probabilistic moment (for \( k > 2 \))

\[
\mu_k(C_{ijkl}^{(p)}) = \int_{-\infty}^{\infty} (C_{ijkl}^{(p)} - E[C_{ijkl}^{(p)}])^k p(C_{ijkl}^{(p)}) d\mathbf{x},
\]  

(27)

- skewness and kurtosis

\[
\begin{aligned}
\beta(C_{ijkl}^{(p)}) &= \frac{\mu_3(C_{ijkl}^{(p)})}{\sigma^3(C_{ijkl}^{(p)})} \\
&= \frac{1}{\sigma^3(C_{ijkl}^{(p)})} \left[ \int (C_{ijkl}^{(p)})^3 - 3 \mu_2(C_{ijkl}^{(p)}) \sigma^2(C_{ijkl}^{(p)}) - \mu_1(C_{ijkl}^{(p)}) \sigma^3(C_{ijkl}^{(p)}) \right],
\end{aligned}
\]  

\[
\kappa(C_{ijkl}^{(p)}) = \frac{\mu_4(C_{ijkl}^{(p)})}{\sigma^4(C_{ijkl}^{(p)})} - 3
\]  

(28)

All aforementioned probabilistic characteristics may be determined with the use of stochastic perturbation technique, where the following expansion of the effective tensor is accomplished:

\[
C_{ijkl}^{(eff)}(b) = C_{ijkl}^{(eff)}(b) + b \eta(C_{ijkl}^{(eff)})(b) + \frac{b^2}{2!} \eta^2(C_{ijkl}^{(eff)})(b^2) + \cdots
\]  

(29)

Then, using the above definitions one may derive analytically the formulas adjacent to the first four probabilistic moments and characteristics [3]: the expectation, for instance, is obtained as

\[
E[C_{ijkl}^{(eff)}] = 1 \times C_{ijkl}^{(eff)}(b) + \frac{1}{2!} \eta^2(C_{ijkl}^{(eff)})(b) \times \text{Var}(b) + \frac{1}{4!} \eta^4(C_{ijkl}^{(eff)})(b) \times \text{Var}^2(b) + \cdots
\]  

(30)

Further, let us represent the stochastic effective tensor using the following time series

\[
C_{ijkl}^{(eff)}(\omega, t) = C_{ijkl}^{(eff)}(\omega) - C_{ijkl}^{(eff)}(\omega) t; \quad t \in [0, \infty),
\]  

(31)

where \( C_{ijkl}^{(eff)}(\omega, t) \) and \( C_{ijkl}^{(eff)}(\omega) \) are Gaussian random variables with the specified and bounded first two probabilistic moments representing the initial elasticity tensor components and their ageing velocity. Then, the expected value of this process may be calculated as

\[
E[C_{ijkl}^{(eff)}(\omega, t)] = E[C_{ijkl}^{(eff)}(\omega)] - E[C_{ijkl}^{(eff)}(\omega)] t.
\]  

(32)

The variance for this process is defined here as

\[
\text{Var}[C_{ijkl}^{(eff)}(\omega, t)] = \text{Var}[C_{ijkl}^{(eff)}(\omega)] + \text{Var}[C_{ijkl}^{(eff)}(\omega)] t^2.
\]  

(33)

Then, the basic characteristics as variation, skewness and kurtosis are simply determined as

\[
\begin{aligned}
\alpha(C_{ijkl}^{(eff)}(\omega, t)) &= \sqrt{\text{Var}[C_{ijkl}^{(eff)}(\omega)] + \text{Var}[C_{ijkl}^{(eff)}(\omega)] t^2} \\
\beta(C_{ijkl}^{(eff)}(\omega, t)) &= \frac{\mu_3(C_{ijkl}^{(eff)}(\omega, t))}{\left( \text{Var}[C_{ijkl}^{(eff)}(\omega)] + \text{Var}[C_{ijkl}^{(eff)}(\omega)] t^2 \right)^{\frac{3}{2}}},
\end{aligned}
\]  

\[
\gamma(C_{ijkl}^{(eff)}(\omega, t)) = \frac{\mu_4(C_{ijkl}^{(eff)}(\omega, t))}{\left( \text{Var}[C_{ijkl}^{(eff)}(\omega)] + \text{Var}[C_{ijkl}^{(eff)}(\omega)] t^2 \right)^{\frac{3}{2}}} - 3.
\]  

(34)
As it is clear from above equations, determination of up to the given order partial derivatives of the effective elasticity tensor with respect to the input random variables is demanded in this method. Since analytical form is unavailable because of the discrete character of effective tensor components determination method, we use the approximation numerical technique called Response Function Method, where for the given time moment \( \tau \) it is proposed that

\[
C_{ijkl}^{\text{eff}}(\tau) = \sum_{n=1}^{\infty} A_{ijkl}^n(\tau) (A(\tau))^n + A_{ijkl}^0(\tau),
\]

(37)

The coefficients of this polynomial form are found from the least squares technique after several homogenization problem solutions with varying values of the input random parameter \( b \). We consider for this purpose a set of \( m \) data points \( \left( b(\beta), C_{ijkl}^{\text{eff}}(\beta) \right) \) for \( \alpha, \beta, \gamma, \delta = 1, 2 \), the nonlinear continuous function \( C_{ijkl}^{\text{eff}} = f(b(\alpha)) \) and a curve (approximating function) \( C_{ijkl}^{\text{eff}} = f(b(A_{ijkl})) \), which additionally depends on \( n \) parameters \( A_{ijkl}^j, j = 1, \ldots, n \), where \( m > n \). The basic difference to the traditional least squares technique is the presence of the fourth order tensors, which are the main goal of this approximation procedure. We define the additional residuals \( r_i \left( C_{ijkl}^{\text{eff}} \right) \) as

\[
r_{ijkl}(\alpha) = r_i \left( C_{ijkl}^{\text{eff}} \right) = C_{ijkl}^{\text{eff}}(\alpha) - f(b(\beta), A_{ijkl}(\beta))
\]

(38)

to generally compute the values of \( A_{ijkl}^j \) from the minimization procedure resulting in the algebraic equations system:

\[
\left( D_{ijkl}^{\alpha} \right)^T w D_{ijkl}^{\alpha} A_{ijkl} = (A_{ijkl}^{\alpha})^T w C_{ijkl}^{\text{eff}}
\]

(39)

where

\[
D_{ijkl}^{\alpha} = \frac{\partial f \left( b(\beta), A_{ijkl}(\beta) \right)}{\partial A_{ijkl}^j}, \quad j = 1, \ldots, n; \quad \alpha, \beta, \gamma, \delta = 1, 2.
\]

(40)

and, where \( w \) denotes the vector of the weights, assumed here for simplicity as the unit vector. This method is used for the few discrete time moments to compute stochastic fluctuations of the effective elasticity tensor. Having determined time fluctuations of the effective elasticity tensor we consecutively determine symbolically partial derivatives of this tensor and use them in the perturbation-based homogenization method, which is used alternatively instead of the Monte-Carlo technique in the view of comparable efficiency and significant time savings [4].

5. Computational analysis

5.1. Probabilistic analysis

Let us consider for an illustration a composite with the square periodicity cell having unit external dimensions; the fiber has a round cross-section and the mean value of its radius is taken as

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic characteristics of the metallic composites.</td>
</tr>
<tr>
<td>No.</td>
</tr>
<tr>
<td>-----</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
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<tr>
<td>3</td>
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<tr>
<td>4</td>
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<td>5</td>
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<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
</tbody>
</table>

Fig. 2. Finite Element Method discretization of the RVE quarter.

Fig. 3. Coefficients of variation of the effective tensor vs. CoV for Young modulus.

Fig. 4. Coefficients of skewness of the effective tensor vs. CoV for Young modulus.
The expected values of the effective elasticity tensor are shown in Table 2 (expected values at the initial stage are taken once more from Table 1, while time rate statistical parameters are defined here as $E(\hat{e}(\hat{x};c_0)) = \frac{0.005E(\hat{e}(x;\omega))}{10}$ and $a(\hat{e}(\hat{x};c_0)) = 0.10 \cdot E(\hat{e}(\hat{x};c_0))$ for both composite constituents. The results of these computations are pro-

The coefficients of variation for $C_{1212}^{\text{eff}}$ are all the same for various combinations of the matrices and fibers and all display perfect linear interrelation with respect to the input coefficient of variation of the Young modulus in the matrix (Fig. 3). Moreover, they are equal to each other, so that one may expect that the variances of effective elasticity tensor increase parabolically together with the random dispersion at the input, as it was documented in [5] for those composites. This information is important taking into account further analysis of composites structures, where the character of random uncertainties within the effective tensor (without a prior mathematical knowledge) may be restricted to some numerically driven properties. Equality of the input and output coefficients of variation automatically leads to the conclusion that the output random variables are Gaussian, which is entirely confirmed in Figs. 4 and 5. Considering a brevity of the presentation we attach the additional moments only for two composites – NiAl–Cr and MoSiO2–SiC. Those two composites having after Table 2 the largest expectations should exhibit largest variations of higher order statistics. As it is clear from both figures, both $\beta(C_{1212}^{\text{eff}})$ as well as $\kappa(C_{1212}^{\text{eff}})$ are very close to 0 and rather independent from $x(E_2)$, so one can conclude that those distributions are Gaussian; the rest of tested composites behave the same for those two characteristics. The conclusion about Gaussian distribution of the effective elasticity tensor for metallic composites is important, because statistical information restricted to the first two statistical moments only gives full knowledge about the uncertainty of those composites effective properties. It significantly shortens the time of computations and allows for the usage of fast computational techniques like the generalized stochastic perturbation method presented here.

### 5.2. Stochastic ageing simulation

The same geometry is analyzed and subjected to the homogenization process as in the previous numerical illustration, however now we consider a linear stochastic time variations of the Young moduli of both components according to Eq. (8). The expected values at the initial stage are taken once more from Table 1, while time rate statistical parameters are defined here as $E(\hat{e}(\hat{x};c_0)) = \frac{0.005E(\hat{e}(x;\omega))}{10}$ and $a(\hat{e}(\hat{x};c_0)) = 0.10 \cdot E(\hat{e}(\hat{x};c_0))$ for both composite constituents. The results of these computations are pro-

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**Table 2**

The expected values of the effective elasticity tensor.

<table>
<thead>
<tr>
<th>No.</th>
<th>Composite</th>
<th>$C_{1111}^{\text{eff}}$ (kPa)</th>
<th>$C_{1212}^{\text{eff}}$ (kPa)</th>
<th>$C_{1617}^{\text{eff}}$ (kPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Mg–C</td>
<td>7.9539E+7</td>
<td>3.1012E+7</td>
<td>2.5541E+7</td>
</tr>
<tr>
<td>2</td>
<td>Al–C</td>
<td>9.8369E+7</td>
<td>3.8631E+7</td>
<td>3.0096E+7</td>
</tr>
<tr>
<td>3</td>
<td>Ti–Al2O3</td>
<td>2.5698E+8</td>
<td>9.5218E+7</td>
<td>1.0449E+8</td>
</tr>
<tr>
<td>4</td>
<td>Ti–SiC</td>
<td>2.6240E+8</td>
<td>9.5218E+7</td>
<td>1.0449E+8</td>
</tr>
<tr>
<td>5</td>
<td>Ti–B</td>
<td>2.6592E+8</td>
<td>9.5218E+7</td>
<td>1.0449E+8</td>
</tr>
<tr>
<td>6</td>
<td>NiAl–Cr</td>
<td>2.6592E+8</td>
<td>9.5218E+7</td>
<td>1.0449E+8</td>
</tr>
<tr>
<td>7</td>
<td>MoSiO2–SiC</td>
<td>4.8525E+8</td>
<td>1.4850E+8</td>
<td>1.6885E+8</td>
</tr>
</tbody>
</table>
Provided in Figs. 6–13, each time for the time interval $t \in [0, 20 \text{ years}]$ and taking into account various levels of random dispersion of the initial Young modulus, i.e. $\alpha = 0.05, 0.10, 0.15$ and 0.20. These results are limited to the two composites only – MoSiO$_2$–SiC and Ti–SiC, because the first shows the highest and second – the intermediate values of the effective elasticity tensor components.

A comparison of time fluctuations for the expected values and coefficients of variation for both composites shows almost the same tendencies, independent from both composite type and the initial input random dispersion of the metallic matrix Young modulus (expectations only). The expected values systematically decrease in time, mostly linearly, in almost the same way as this Young modulus is stochastically reduced. A total effective reduction of all components for the homogenized tensor is around 10% of their initial values for the time interval of 20 years (assuming the mean reduction of 0.5% for this modulus per a year). The coefficients of variation time fluctuations highly depends on the time variations of the Young modulus of the metallic phase – of course this is direct proportionally – the higher input coefficient, the higher output uncertainty. As one could expect from the tests provided

Fig. 7. The expected values (in Pa) and coefficients of variation for $C_{1111}^{\text{eff}}(\alpha; t)$ [MoSiO$_2$–SiC].

Fig. 8. The expected values (in Pa) and coefficients of variation for $C_{1122}^{\text{eff}}(\alpha; t)$ [MoSiO$_2$–SiC].

Fig. 9. The expected values (in Pa) and coefficients of variation for $C_{1111}^{\text{eff}}(\alpha; t)$ [Ti–SiC].
for the carbon-epoxy composites [4], now the uncertainty level in homogenized tensor increases together with time being and this increase is also almost linear (some typically nonlinear behavior is observed for $C_{\text{eff}}(\alpha, t)$ in Ti–SiC composite). However, it can be noticed that the higher initial value of this coefficient, the larger increase in the specified time interval, so that output randomness level is at least the few percents higher than the input one and it agrees with engineering intuition, where the materials subjected to extensive ageing show a decrease of elastic characteristics and an increase of the structural uncertainty. Contrary to the previous experiments results, now the effective elasticity tensor appears to be non-Gaussian; skewness and kurtosis both depend on the input coefficient of variation of $E_2$. Now, the skewness decreases from some positive values smaller each time than one to the values very close to 0 – we notice that the higher input coefficient of variation, the larger values of this skewness' coefficient. Those time fluctuations are also quasi-linear (as for $\alpha C_{\text{eff}}(\alpha, t)$) and the largest decrease is observed for the largest value of the input coefficient of variation (in the range of almost 20%). The time fluctuation of kurtosis is the
most significant in-between all the basic probabilistic moments – from the value around 0 it decreases to almost –1.6 (for MoSiO₂–SiC) or even –1.2 (for Ti–SiC). They are also decisively less dependent on the input coefficient of variation – contrary to all previous cases now largest sensitivity of the kurtosis is observed for MoSiO₂–SiC (maximum values of the homogenized tensor) than for Ti–SiC (intermediate values). Independently from the composite type the final distribution of the effective elasticity tensor becomes apparently non-Gaussian one with decreasing expectations and increasing random dispersions (more less symmetrically distributed around the additional expectations). Let us note that the time period of 20 years is not the entire exploitation time for those materials, rather their beginning stage, but the observed tendencies may allow for their time-dependent reliability prognosis. Since the nature of the metallic composites collected here, the corrosion processes in their volumes and their surfaces may be the main source of this ageing, so that the rules noticed here may have a dominant role.

6. Concluding remarks

As it is documented by the numerical tests provided in the paper, the effective elasticity tensor components have Gaussian distribution for the metallic composites with Gaussian distribution of the Young moduli (for the fiber and matrix both and separately). Usually, the larger is metallic component modulus mean value, the larger expectation of the effective elasticity tensor, and the coefficients of variation for this tensor are exactly equal to the random dispersion of the input Young modulus, while the remaining characteristics are equal to 0. The time fluctuations of those moments, according to the simplest linear stochastic decay form, show that the expectations decrease linearly (in the range of the few percent), coefficients of variations increase (almost in the same range), while the skewness and kurtosis generally decrease both however with different rates per a year. Considering above, together with the so defined ageing process (having its justification in the engineering practice and evidence) the effective elasticity tensor for the metallic composites becomes apparently non-Gaussian. Considering above, at the initial stage of the exploitation we can use the first two probabilistic moments only to recover all uncertainty in the homogenization, but afterwards the precise reliability analysis would demand higher order statistics also [29]. The presented hybrid probabilistic computational methodology may be further expanded from the linear elastic time-independent and dependent problems towards the elasto-plastic behaviour of metallic composites using both commercial and academic software, like the computer software described in [6]. A comparison against the other probabilistic and stochastic methods available in the area of homogenization method would be valuable considering computational power and time consumption [10].

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