NUMERICAL SIMULATION OF THE EULER PROBLEM FOR THE ELASTIC BEAMS WITH RANDOM PARAMETERS

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ABSTRACT: The main aim is to present the combined Response Function Method with the Stochastic Finite Element Method to study the probabilistic moments of the critical force in linear elastic buckling analysis. This approach illustrated by the classical Euler problem of elastic beam having Gaussian random length shows very good accuracy in a comparison with the classical Monte-Carlo simulation. The technique demonstrated does not need a direct differentiation of all system matrices and the solution of up to nth order equations during finite element analysis. It demands traditional FEM software together with polynomial approximation technique implemented in the computer algebra system MAPLE to recover all necessary probabilistic moments for the critical force. Various aspects of this hybrid FRM-SFEM approach are studies here like the probabilistic convergence, effectiveness of the polynomial approximation and its influence on the probabilistic moments recovered and all those studies confirm the usefulness of the technique invented in further, not only linear buckling, engineering problems.

Key words: critical force, Stochastic Finite Element Method, response function method, Monte-Carlo simulation, buckling analysis

1. INTRODUCTION

Stochastic or probabilistic buckling of the engineering structures is still an important research area since many of those structures exhibit unpredictable fluctuations of the material and geometrical parameters [11]; they can be also subjected to the stochastic loads [2]. Those variations, sometimes of apparently local character, may significantly change the critical force as one may guess on the basis of the Euler formula. The stability theory has still many open questions [4,5,7], even in the area of elastic beams, so that probabilistic analytical approaches are not of the very general character and, furthermore, are not available for more complex problems like buckling of plates and/or shells with not trivial boundary conditions. The second reason to deal with this problem is the that the critical behavior is decisive for many structures that need to be optimally designed and, as it is documented by the engineering practice is frequently not accounted for. Since the new Eurocodes for engineering design introduce a necessity of probabilistic design and reliability index determination, there is no doubt that this research area may be important in at least civil engineering. There are the several numerical methods to analyze the buckling phenomenon for the engineering systems with random parameters [3,6]. Besides the simulation techniques like crude or weighted Monte-Carlo scheme (MCS) or Metropolis approaches, the spectral methods as well as the hybrid semi-analytical approaches one may find the whole family of perturbation methods [1]. The main value of this last class for the engineers is that the computational time is relatively short, there is no need for massive computers to provide the analysis and that they are relatively easy implementable into any computer system [8,9]. The quality of the perturbation method strongly depends on the user programming skills and its will patience to include as many higher order terms as possible. In the view of above, the main aim of this paper is to show the application of the generalized nth order perturbation method implemented into the academic FEM software to find the probabilistic moments for the Euler critical force. This relatively easy task is performed to validate the entire theory against the MCS results obtained for the analytical formula widely available and valid for the elastic beam with constant prismatic cross-sectional area under small deflections. The integral part of the computational tools employed in this work is the computer algebra system MAPLE, which is used for both simulation purposes as well as for the processing of the probabilistic characteristics and automatic differentiation in the perturbation approach. The second, not less important reason, is a verification of the Stochastic Finite Element Method [8-10] convergence in the traditional Finite Element Method context – by increasing the basic element length of the uniform mesh. The very good accuracy obtained in both cases, for the meshes from $2^2$ to $2^7$ finite elements and for large random dispersion of the input uncertain Young modulus prove the quality of the innovative Response Function Method co-implemented with the generalized SFEM. This technique does not require straightforward differentiation of equilibrium equations and their further solution but enables for direct determination of analytical polynomial interrelation between the chosen structural response and given random input. This approach is of a special value since its wide applicability in nonlinear problems of modern computational mechanics and engineering.
2. ELEMENTS OF STATISTICAL ANALYSIS

The applied descriptive statistics for the point distributions considered here is based on N computational experiments, where the critical force value \( \mu_{cr}^{(i)} \) for \( i=1,\ldots,N \) is determined. Statistical estimation theory defines in this context the expected value of this force as

\[
E[P_{cr}] = \frac{1}{N} \sum_{i=1}^{N} P_{cr}^{(i)}. 
\]

(1)

Further, the unbiased estimator for the variance is applied in the form

\[
Var(P_{cr}) = \frac{1}{N-1} \sum_{i=1}^{N} (P_{cr}^{(i)} - E[P_{cr}])^2. 
\]

(2)

which enables to determine the standard deviation as its square root

\[
\sigma(P_{cr}) = \sqrt{Var(P_{cr})}. 
\]

(3)

The ratio of the standard deviation to the expected value is called the coefficient of variation

\[
\alpha(P_{cr}) = \frac{\sigma(P_{cr})}{E[P_{cr}]}.
\]

(4)

Generally, the \( m \)th central probabilistic moment is estimated using the following formula:

\[
\mu_m(P_{cr}) = \frac{1}{N} \sum_{i=1}^{N} (P_{cr}^{(i)} - E[P_{cr}])^m. 
\]

(5)

which practically is most frequently used to approximate third and fourth central probabilistic moments. Although the values of the latters equal 0 and 3, respectively, for the Gaussian variables, real computations prove that one needs the very large populations to obtain those results with the satisfactory precision. We additionally define the asymmetry coefficient and kurtosis as

\[
\beta(P_{cr}) = \frac{\mu_3(P_{cr})}{\sigma^3(P_{cr})}, \quad \kappa(P_{cr}) = \frac{\mu_4(P_{cr})}{\sigma^4(P_{cr})} - 3. 
\]

(6)

There is no doubt that the unbiased estimators for higher moments have rather complex algebraic structures; as the example one may consider the kurtosis as

\[
\kappa(P_{cr}) = \frac{N(N+1)}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N} \left( \frac{P_{cr}^{(i)} - E[P_{cr}]}{\sigma(P_{cr})} \right)^4 \frac{3(N-1)^2}{(N-2)(N-3)}. 
\]

(7)

Sometimes, as far as the analytical functions are available, one may verify the correctness of the statistical estimation with the analytical derivations for probabilistic moments. Especially attractive are the polynomial forms, where one can apply for the Gaussian variable \( X \) with mean value \( E[X] \) and standard deviation \( \sigma(X) \) the following rules:

\[
\begin{align*}
E[X] &= \sigma^2(X) + E^2[X] \\
E[X^2] &= 2\sigma^2(X)[E[X] + E^2[X]] + E^4[X] \\
E[X^3] &= 3\sigma^3(X)[E^2[X] + E^3[X]] + 6\sigma^2(X)[E^2[X] + E^3[X]] + E^4[X] \\
E[X^4] &= 3\sigma^4(X) + 6\sigma^3(X)[E^2[X] + E^3[X]] + 4\sigma^2(X)[E^2[X] + E^3[X]] + E^4[X]
\end{align*}
\]

(8)

which finally give

\[
\mu_3(X) = 3\sigma^4(X) 
\]

(9)

and which all follow simple integration from the additional definitions.

In the analysis we have made an assumption that Euler function (1) was random variable where parameter of random was length of beam \( L \).

\[
P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 Ebh^3}{12L^2}
\]

(10)

The Monte-Carlo simulations were realized for normal distribution with probability density function (2)

\[
f(L) = \frac{1}{\sqrt{2\pi\sigma(L)^2}} \exp \left( -\frac{(L - E[L])^2}{2\sigma^2(L)} \right)
\]

(11)

The statistic tests were realized for the sample size \( N=10^6 \) and all the solutions were presented for critical force \( P_{cr} \) with the following parameters:

- mean \( E[L]=3.3809 \) and standard deviation \( \sigma(L)=0.012 \).
3. THE STOCHASTIC PERTURBATION METHOD

Let us introduce the random variable \( b = b(\omega) \) and its probability density function as \( p(b) \). Then, the expected values as well as its central nth probabilistic moments are defined as

\[
E[b] = b^0 = \int_{-\infty}^{\infty} b p(b) \, db ,
\]

and

\[
\mu_n(b) = \int_{-\infty}^{\infty} (b - E[b])^n \, p(b) \, db .
\] (13)

The basic idea of this stochastic perturbation approach is to expand all the input variables and all the state functions of the considered problems via Taylor series about the additional expected values using the parameter \( \varepsilon = 0 \). In case of random quantity \( b = \varepsilon \), the following expression is employed:

\[
e = \varepsilon^0 + \sum_{n=1}^{\infty} \frac{1}{n!} \varepsilon^n \frac{\partial^n E[b]}{\partial \varepsilon^n} \Delta(b)^n,
\] (14)

where

\[
\Delta(b) = b - b^0
\] (15)
is the first variation of \( b \) around its expected value \( b^0 \). The symbol \( (\cdot)^n \) will denote further the first partial derivative with respect to \( b \) evaluated at \( b^0 \), respectively. Let us consider some random function \( f(b) \) and let us calculate the expected value of by expanding according to formula (14) and inserting into the definition (12). There holds

\[
E[f(b);b] = \int_{-\infty}^{\infty} f(b) p(b) \, db = \int_{-\infty}^{\infty} \left[ f^0 + \sum_{n=1}^{\infty} \frac{1}{n!} \varepsilon^n \frac{\partial^n E[f(b)]}{\partial \varepsilon^n} \Delta(b)^n \right] p(b) \, db
\] (16)

Let us remind that this power expansion is valid only if the state function is analytic in \( \varepsilon \) and the series converges and, therefore, any criteria of convergence should include the magnitude of the perturbation parameter; perturbation parameter is usually taken simply as equal to 1 in engineering computations. Contrary to most of previous analyses in this area, now the parameter \( \varepsilon \) is treated as the expansion parameter in further analysis, so that is included explicitly in all the further derivations demanding analytical expressions. Numerical studies will show the influence of this parameter on the up to 4th central probabilistic moments of the critical force and will be obtained as the polynomials of the additional order with respect to the input coefficient of variation \( \Delta(b) \) as well as the perturbation parameter \( \varepsilon \).

From the numerical point of view, the expansion provided by the formula (14) is carried out for the summation over the finite number of components, whereas the integrals given in definitions (12,13) are never calculated with infinite limits – usually they have lower and upper bounds driven by physical meaning of the specific parameter or just the experimental works (like 0 at least for the beam length). Considering various probability distributions, the essential difference is noticed between symmetric distribution functions, where one uses

\[
E[f(b);b] = f^0 + \sum_{n=1}^{\infty} \frac{1}{n!} \varepsilon^n \frac{\partial^n E[f(b)]}{\partial \varepsilon^n} \Delta(b)^n p(b) \, db
\] (17)

and non-symmetric probability functions

\[
E[f(b);b] = f^0 + \int_{-\infty}^{\infty} \frac{1}{n!} \varepsilon^n \frac{\partial^n E[f(b)]}{\partial \varepsilon^n} \Delta(b)^n p(b) \, db
\] (18)

Let us focus now on analytical derivation of the first four probabilistic moments for the structural response function related to some Gaussian input \( b \). The expected value is calculated from its definition as

\[
E[f(b);b] = f^0(b) + \frac{1}{2} \varepsilon^2 f_{bb}(b) \mu_2(b) + \frac{1}{6} \varepsilon^3 f_{bbb}(b) \mu_3(b) + \frac{1}{24} \varepsilon^4 f_{bbbb}(b) \mu_4(b) + \ldots
\] (19)

This expected value can be calculated analytically or symbolically computed only if it is given as some analytical function of the random input parameter \( b \); many existing models in various branches of civil engineering (like the basic rules originated from the strength of materials) can be adopted to achieve this goal. Computational implementation of the symbolic calculus programs (with automatic partial differentiation of even complex real functions), combined with powerful visualization of probabilistic output moments, assures the fastest solution of such problems. Further, thanks to such a series representation of the random output, any desired efficiency of the expected values as well as higher probabilistic moments can be achieved by an appropriate choice of the expansion length and some additional correction available in the parameter \( \varepsilon \), which depend on the input probability density function (PDF) type, interrelations between the probabilistic moments, acceptable error of the computations etc. This choice can be made by comparative studies with sufficiently long (almost infinite) series Monte-Carlo simulations or theoretical results obtained from the direct symbolic integration. Similar considerations lead to the 6th order expressions for a variance; there holds

\[
Var(f) = \int_{-\infty}^{\infty} f^0 + \varepsilon f_{b}(b) \, db + \frac{1}{2} \varepsilon^2 f_{bb}(b) \Delta(b)^2 + \frac{1}{6} \varepsilon^3 f_{bbb}(b) \Delta(b)^3 + \frac{1}{24} \varepsilon^4 f_{bbbb}(b) \Delta(b)^4 - E(f)^2 \, p(f(b)) \, db
\] (20)

As it can be recognized here, the first integral corresponds to the second order perturbation, the next three complete 4th order approximation and the rest needs to be included to achieve full 6th order expansion, so that those formulas may be implemented as the equations of the recursive type. After a multiple integration and the upper indices transformations, one can show that

\[
Var(f) = \varepsilon^2 \mu_2(f) f_{b}(b) + \varepsilon^4 \mu_4(f) \left( f_{bb}(b) \right)^2 + \frac{1}{2} \varepsilon^6 \mu_6(f) \left( f_{bbbb}(b) \right)^2
\] (21)

Quite similarly, using the lowest order expansions, it is possible to derive third central probabilistic moments as

\[
\mu_3(f(b);b) = \int_{-\infty}^{\infty} \frac{1}{2} \varepsilon^3 f_{bb}(b) \Delta(b)^3 p(b) \, db
\] (22)

as well as fourth central probabilistic moment in the form of

\[
\mu_4(f(b);b) = \int_{-\infty}^{\infty} \frac{1}{6} \varepsilon^4 f_{bbb}(b) \Delta(b)^4 p(b) \, db
\] (23)

Let us mention that it is necessary to insert in each of those equations the relevant central probabilistic moments of the input random variables to get the algebraic form convenient for any symbolic computations. A recursive derivation of the particular perturbation order equilibrium equations can be powerful in conjunction with symbolic packages with automatic differentiation tools only; it can potentially extend the area of stochastic perturbation technique applications in computational physics and engineering outside the random processes with small dispersion about their expected values. Hence, there is no need to implement directly exact formulas for a particular nth order equations extracted from the perturbation - they can be symbolically generated in the system MAPLE, like it is done here, and next converted to the FORTRAN source codes of the relevant computer software. Finally, it
should be emphasized that the random input variables must express the uncertainty in space or in time, separately. The expansion including the perturbation parameter \( \varepsilon \), a perturbation order \( m \) as well as the input random variable \( b \) for symmetric PDFs looks like

\[
E[f(b), b, \varepsilon, m] = f^0(b) + \varepsilon E \left[ \frac{\partial^2 f}{\partial b^2} \mu_2(b) \right] + \frac{1}{2} \varepsilon^2 E \left[ \frac{\partial^4 f}{\partial b^4} \mu_4(b) \right] + \ldots
\]  

(24)

4. ELASTIC BUCKLING ANALYSIS BY THE FEM

The problem of buckling analysis appeared in the literature as eigenvalue problem with parameter of load \( \lambda \). Let us consider the linear elastic prismatic beam element with two nodes indexed by \( a \) and \( b \) at its ends. The energy of deformation in element can show that

\[
U = \frac{1}{2} \mathbf{q}^T \mathbf{K}^{(e)} \mathbf{q} + \frac{1}{2} \mathbf{q}^T \mathbf{k}^{(e)} \mathbf{q}
\]  

where \( \mathbf{q} \) is the displacement vector of the nodes

\[
\mathbf{q} = \{ q_{x,a}, q_{y,a}, q_{x,b}, q_{y,b} \}
\]

(25)

\( \mathbf{k}^{(e)} \) is the elastic stiffness matrix of element

\[
\mathbf{k}^{(e)} = \frac{E I}{L}
\]

(26)

and \( \mathbf{k}^{(g)} \) - the geometric stiffness matrix of element

\[
\mathbf{k}^{(g)} = \frac{E I}{L}
\]

(27)

Then, the elemental potential energy may be expressed as

\[
J_p = \frac{1}{2} \mathbf{q}^T \left( \mathbf{k}^{(e)} + \mathbf{k}^{(g)} \right) \mathbf{q} - \mathbf{R}^T \mathbf{q}
\]  

(28)

the minimization of which yields

\[
\left( \mathbf{k}^{(e)} + \mathbf{k}^{(g)} \right) \mathbf{q} = \mathbf{R}
\]  

(29)

The discrete stability equation rewritten for the entire system is

\[
\left( \mathbf{K}^{(i)} + \lambda \mathbf{K}^{(g)}(\mathbf{F}^*) \right) \mathbf{r} = \lambda \mathbf{R}^* \quad \mathbf{r}_1 \neq \mathbf{r}_2
\]  

(30)

and next we obtain the basic equation of stability

\[
\frac{\mathbf{K}^{(i)} + \lambda \mathbf{K}^{(g)}(\mathbf{F}^*)}{\lambda} \mathbf{v} = 0
\]  

(31)

\( \Gamma \) denotes the matrix composed from the derivatives of displacement \( u_i \), \( i = 1, 2 \). Therefore, the basic condition that one can get for the critical value \( \lambda = \lambda_c \), and for critical load \( R_c = \lambda_c \mathbf{R}^* \) is the following one:

\[
\text{det} \left( \mathbf{K}^{(i)} + \lambda \mathbf{K}^{(g)}(\mathbf{F}^*) \right) = 0
\]  

(32)

Contrary to the existing stochastic perturbation methods, the Response Function Method requires the few solutions of eqn (35) around the mean value of the chosen random input parameter, which is the subject of further analysis.

5. COMPUTATIONAL ANALYSIS

The first group of numerical tests was devoted to the convergence verification of polynomial approximation typical for the Response Function Method in the context of the traditional Finite Element Method. We tested the beams discretized using 8, 16, 32, 64 and 128 linear 2-noded finite elements, where the beam length was varying in the interval \( P_a, L \in P_a \{1-50%, 50%-90% \} \).

Tab. No. 3. The results for 8 finite elements

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<th>Length [m]</th>
<th>Increment [m]</th>
<th>Total length [m]</th>
<th>Critical force ( P_c ) [N]</th>
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The results of this analysis are presented graphically in Figs. 3 and 4 for 8 and 64 finite elements as well as numerically in Tabs. 3 and 4. Some small differences were noticed but both character and the shape of the response functions determined were perfectly the same. Further, a small difference between critical force value for beam was noticed between 32 and 64 finite elements discretizations and a complete lack of a difference between critical force value obtained for 64 and 128 elements. The difference between critical force value determined from the Euler formula and the FEM for 128 elements was equal to 0.017%. Finally we studied here a dependence of the polynomial approximation on the total number of the sampling points. The approximation curves for 10 and 7 points calculated (the range of critical values $P^c[L]$, $P_r[L^8, 50%, L^4+50%]$ for 128 finite elements meshes have the following form:

\[
y = -0.0050541896x^9 + 0.1489940359x^8 + 1.940822699x^7 + 14.66104847x^6 - 70.77430775x^5 + 226.4115405x^4 - 480.0097605x^3 + 650.4556605x^2 - 178.2053960\]

and

\[
y = 0.0041671850x^8 - 0.085684198x^7 + 0.7332807876x^6 - 3.34889084x^5 + 8.637259203x^4 - 12.01850121x - 7.198696376\]

Let us note that in a general case following an application of Eqn (24) one can also have the perturbation parameter combined with the additional powers for each component separately – as it is documented by further computational results. The diagram of the expected values of the critical force with respect to the coefficient of variation $\alpha \in [0.0, 0.15]$ and mean length $L = 3.3809$ are shown below (see Fig. 5). As one may see, there is no difference between the results obtained for $2^3$, $4^3$, $8^3$ and $10^3$ order perturbation approaches.

A comparison of the expected values obtained for 10 (Fig. 5) and 7 points (Fig. 6) approximations shows that smaller number of the sampling points leads to the overestimation of those expectations. Furthermore, the differences between neighboring stochastic perturbation orders increase also. The same comparison of the variances – for 10 points (Fig. 7) and 7 points (Fig. 8) recovering of the response functions shows quite inverse effect than before – smaller size of this approximation should result in some underestimation of those variances.
Higher moments are not very sensitive – one can compare third probabilistic moments for 10 points approximation (Fig. 9) and for 7 points (Fig. 11) as well as the fourth central moments (Figs. 10 and 12, respectively) to conclude no apparent changes between them. Let us note that the surface representations are given with respect to two independent parameters – the perturbation parameter \( \varepsilon \in [0.8, 1.2] \) and the coefficient of variation of the random length \( \alpha \in [0.0, 0.2] \).

Next, we adopted perturbation parameter \( \varepsilon \) equal to 1 to get more numerical comparison (this value is traditionally used in all computations). Then, the largest difference between 6th order stochastic perturbation result for the variance (see Figs. 13 and 14) for \( \alpha=0.2 \) was about 25\%.
Considering the numerical discrepancies discussed above, the approximation basis has been finally increased to 15 points determined with the use of 128 finite elements once more and for $P_c(L)=P_c[L_0-50\%,L_0+50\%]$. The polynomial curve which traversed the chosen points was obtained as

$$y = -0.0000128984x^7 + 0.0005770210x^6 - 0.00099034777x^5 + 0.0901633513x^4 - 0.4688180626x^3 + 1.435977488x^2 - 2.470167812x + 1.948420351$$

(30)

which resulted in Fig. 15; the trial points are marked here with the diamonds. The expected values resulting from this approximation are graphically presented in Fig. 16 and it is clear that no significant difference to the 10 points approximation is noticed (cf. Fig. 5). Then we have in turn the variances according to 2nd, 4th and 6th order perturbation formulas (Fig. 17), third (Fig. 18) and fourth (Fig. 19) central probabilistic moments; they are all the functions of the independent parameters $\varepsilon \in [0.8,1.2]$ and $\alpha \in [0.0,0.2]$. 

Fig. 13. Variance for 10 points approximation

Fig. 14. Variance for 7 points approximation

Fig. 15. Polynomial approximation for 15 points

Fig. 16. The expected values for 15 points approximation

Fig. 17. Variance for 15 points approximation
Finally, we have a variability of the variances for $\varepsilon=1$ and 15 points polynomial approximation in Fig. 20, which contrasted with Figs. 13 and 14 shows that the variances converge asymptotically together with the number of sampling points in the response function approximation.

Finally, a validation of the Stochastic Finite Element Method based on the Response Function Methods against the Monte-Carlo simulation (based on the analytical equation for the critical force) has been performed in details. The results are presented in Fig. 21 for expected values, in Fig. 22 for third central probabilistic moments, in Fig. 23 for fourth central probabilistic moments and in the case of variances – in Fig. 24. The results of the Monte-Carlo simulation performed in the crude version using the system MAPLE are marked with the diamonds on all those graphs; the expected values were computed here using $2^{\text{nd}}$, $4^{\text{th}}$, $6^{\text{th}}$, $8^{\text{th}}$ and $10^{\text{th}}$ order stochastic perturbation expansions, whereas the variances – with the use of up to $6^{\text{th}}$ orders only. How it is shown in Figs. 21 and 24, the difference between the MCS and the SFEM decreases together with an increase of the perturbation order (even for larger coefficients of variation). Higher central moments are computed with satisfactory accuracy in the range of $\alpha \in [0.0,0.15]$ and for larger values of this parameter they simply need longer expansions than those provided by the formulas (22) and (23).
6. CONCLUDING REMARKS

The Stochastic Finite Element Method analysis and the Monte-Carlo simulation of the elementary Euler problem have been performed in this paper. As it was proved by the computational analysis, the generalized stochastic perturbation technique implemented in the Response Function Method version into the Finite Element Method is very accurate, since even third and fourth central probabilistic moments are very close to the MCS results for the random dispersion range [0.0, 0.2] obeying most of practical engineering applications. It is also apparent that the hybrid symbolic-FEM implementation with the use of the computer algebra system MAPLE, v. 11 is the very effective numerical tool – all probabilistic characteristics may be determined as the functions of the perturbation parameter ε, which was quite impossible before. As it was expected on the basis of the previous computational experiments, the method converges rather fast together with an increase of the stochastic perturbation order and for 10th order analysis the differences practically are invisible. The Response Function Method appeared to be almost independent on the number of the trial points taken initially around the mean value to provide the additional approximation. Finally, the entire methodology presented may be directly used in the reliability analysis for the structures protected against the buckling, to provide their optimal design.

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8. REFERENCES